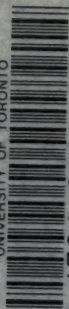



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OF
TIME AND SPACE

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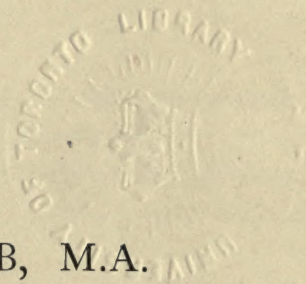
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A THEORY OF TIME AND SPACE

BY

ALFRED A. ROBB, M.A.



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1914

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A THEORY

OF TIME AND SPACE



“Ὁ ἐμὲ γλῶσσε θεέ, ἀγύρ' ὀνόμα' να ἠέμεανν.”

1911
P. 10/11

PREFACE

THE introduction to the present volume was first published in 1913*, in substantially the form here reproduced.

The plan there outlined is here carried out in detail, and, although much remains to be done along similar lines, yet the present work is fairly complete in itself.

The special object here aimed at has been to show that spacial relations may be analyzed in terms of the time relations of *before* and *after*, and the demonstration of this thesis has been carried out up to the stage of introducing coordinates, leaving further developments for a future volume.

The present work is the outcome of an endeavour to get rid of certain obscurities in connection with some of the fundamental parts of Physical Science.

Thus the meaning ordinarily attached to the equality of time intervals and of lengths, although sufficiently precise for the ordinary purposes of daily life, will yet be found to be somewhat vague when examined more closely.

Again, although the idea of acceleration plays such an important part in Dynamics, yet, so far as the writer is aware, no satisfactory definition has hitherto been given of what we mean when we say that a particle is "unaccelerated."

All attempts in this direction tacitly assumed that we had already at our disposal some unaccelerated body or system to which the motions of other bodies or systems might be referred.

* *A Theory of Time and Space*, Heffer and Sons, Cambridge, 1913.

The present work aims at giving precision to ideas of this kind by reducing them to more fundamental concepts.

The method here pursued leads to the construction of a system of Geometry having many curious characteristics, and of which ordinary Euclidean Geometry forms a part.

Although the use of figures is of great assistance in this as in other types of Geometry, yet it is to be remembered that they are here merely aids to the imagination, and it has been deemed advisable, in the majority of cases, to leave the construction of them to the reader.

In conclusion I desire to express my best thanks to Professor Sir Joseph Larmor, whose kindly encouragement, given from time to time, has done much to counteract those occasional feelings of discouragement which are perhaps inseparable from work of this sort.

I further desire to express my thanks to the officials of the University Press for their never-failing courtesy and for the care and trouble which they have taken in the production of this book.

ALFRED A. ROBB.

CAMBRIDGE,

September 19, 1914.

A THEORY OF TIME AND SPACE

INTRODUCTION

In the following pages the writer proposes to give an account of an investigation of the relations of Time and Space in connection with the physical phenomena of Optics.

The subject is thus in part philosophical, in part mathematical, and in part physical.

Under the name of "The Theory of Relativity" this subject has been much under discussion, but it is still in a condition of considerable obscurity.

Although generally associated with the names of Einstein and Minkowski, the really essential physical considerations underlying the theories are due to Larmor and Lorentz.

According to the Newtonian mechanics there is no perceptible distinction between a system of bodies "at rest" and one moving with uniform velocity in a straight line. The velocity is only apparent when two distinct bodies are compared, and, so far as the mathematics is concerned, it is a matter of indifference which of the two be regarded as "at rest."

It remained to be considered whether the phenomena of Optics and Electricity might not afford some means of distinguishing between the two bodies in this respect, but experiment failed to show any.

It was then shown by Larmor that the electromagnetic equations could, by a linear substitution, be made to assume the same form when taken with respect to a system moving with uniform velocity as they had when taken with respect to a system "at rest," and similar results were arrived at by Lorentz. This seemed to indicate that, even if such a thing as "absolute rest" did exist, we should not be able in this way to distinguish it from motion with a uniform velocity.

The question thus naturally arose as to whether any real distinction existed; since philosophers had long contended that all motion was relative.

The subject was rendered more complex and difficult to grasp by the circumstance that, in passing from a system "at rest" to one moving with uniform velocity, a "local time" had to be introduced, and further, bodies appeared to contract in the direction of their motion.

The question of "local time" appeared the greatest obstacle to an acceptance of the view that velocity is merely a relative phenomenon.

We are all familiar with the use of "local time" at different places on the earth's surface, but the two cases are not analogous.

Although noon in Greenwich and noon in New York are both represented as twelve o'clock local time, yet no one would contend that noon in Greenwich is *at the same instant* as noon in New York.

In the case, however, of two material systems which were moving with uniform velocity relative to one another, events which were regarded as "simultaneous" in the one system according to the one "local time" could not, in general, be regarded as "simultaneous" according to the "local time" of the other system. If one of these "local times" were taken as the *true time* and the other regarded as a mathematical fiction, the two systems could not be considered as exactly on a par with one another, and the motion could not be a purely relative phenomenon.

Thus on the one hand, the mathematics seemed to suggest that either of the systems might be regarded as "at rest," while considerations as to the "simultaneousness" of events, on the other hand, appeared to introduce dissymmetry.

It was in order to preserve symmetry that Einstein made the suggestion that events might be simultaneous to one observer, but not simultaneous to another. This remarkable suggestion was at once seized upon, without it apparently being noticed that it struck at the very foundations of Logic. That "a thing cannot both be and not be *at the same time*" has long been accepted as one of the first principles of reasoning, but here it appeared for the first time in science to be definitely laid aside, and although many of those who accepted Einstein's view saw that there was something which was psychologically very strange about it, yet this was allowed to pass in view of the beauty and symmetry which seemed, in this way, to be brought about in the mutual relations of material systems. To others, however, this view of Einstein's appeared too difficult to grasp or analyse, and to this group the writer must confess to belong.

Much of the subsequent development, such as that of Minkowski, has been of a purely analytical character, while the philosophical

difficulty seems to remain much in the same state as it was left by Einstein.

In 1911, the writer published a short tract entitled "Optical Geometry of Motion, a New View of the Theory of Relativity*," in which was put forward an outline of a method of treatment in which he avoided any attempt to identify instants of time at different places. The view was advanced that the axioms of Geometry might be regarded mostly as the formal expression of certain optical facts. As was remarked in the preface to this tract, there are more aspects of things than one, and the fact that we give a physical signification to axioms of geometry, by no means implies that we do not regard their logical aspect as of equal importance.

As regards the physical significance, there is, of course, also more than one; but it appears desirable that we should give the axioms, so far as is possible, an optical significance primarily, rather than have the significance of some of them an optical one, while that of others is made to depend upon the properties of so-called rigid bodies, &c. This is particularly desirable because we have to consider what appears to be a contraction in the line of motion of a body which moves relatively to another, and the question naturally arises how is such contraction to be measured?

How, too, is velocity to be measured, since velocity depends upon distance as well as time? The whole subject is one which can bristle with circular definitions, which, in fact, are very difficult to escape, since ideas which we have been in the habit of regarding as fundamental are here called in question; and the only way out of the difficulty appears to be that we should make a careful re-examination of these apparently fundamental ideas so as to analyse those which are really complex into their separate components.

It was with this object in view that the work here briefly sketched was undertaken.

In the above-mentioned tract this analysis was made to a certain extent; but a number of logical details were omitted, and, since its publication, the author has devised a new method of approaching the problem, which illuminates certain points that were formerly obscure.

This method involves an idea which is believed to be new, and which may be shortly described as the idea of *Conical Order*. This idea we shall now proceed to present in a general way and point out its connection with our problem.

The systematic working out of the idea by means of a series of

* Heffer and Sons, Cambridge, 1911.

postulates and definitions will form the main subject of a detailed discussion, and we are at present only concerned with an attempt to convey the general point of view.

An element of time is called an instant and is to be regarded as a fundamental concept.

Of any two elements of time of which I am *directly* conscious one is *after* the other.

The relation of two instants, one of which is *after* the other, is an asymmetrical relation, and the converse asymmetrical relation is denoted by the term *before*, so that if an instant *A* be *after* an instant *B*, the instant *B* is *before* the instant *A*.

The set of instants of which I am directly conscious form a series in linear order. Thus they satisfy the following conditions:

(1) If an instant *A* be *after* an instant *B*, the instant *B* is not *after* the instant *A*, and is said to be *before* it.

(2) If *A* be any instant, I can conceive of an instant which is *after A* and also of one which is *before A*.

(3) If an instant *A* be *after* an instant *B*, I can conceive of an instant which is both *after B* and *before A*.

(4) If an instant *B* be *after* an instant *A* and an instant *C* be *after* the instant *B*, the instant *C* is *after* the instant *A*.

(5) If an instant *A* be neither *before* nor *after* an instant *B*, the instants *A* and *B* are identical.

If, now, we examine the fifth of these conditions, it might perhaps be thought that it was a necessary consequence of our conceptions of *before* and *after*.

✱ That it is in reality no logical consequence of the other conditions may be shown by the help of a geometrical illustration. This illustration is suggestive, but the development of our theory is in no logical sense dependent upon it.

Let us consider a system of cones having their axes parallel and having equal vertical angles.

Let us regard any cone of the set as terminating in the vertex and as having the opening pointed upwards, let us say. We may call such a cone an α cone, and one with the opening pointed downwards a β cone. The vertex in either case is to be regarded as belonging to the respective cones.

Thus, corresponding to any point of space, there is an α cone of the

set having that point as vertex, and similarly there is also a β cone of the set having the point as vertex.

Now it is possible, by using such cones and by making a certain convention with respect to the use of the words *before* and *after*, to set up a type of order of the points of space.

If A_1 be any point and α_1 the corresponding α cone, then we shall say that any point A_2 is *after* A_1 , provided that the points are distinct and A_2 lies either on or inside the cone α_1 .

Similarly, if A_1 be any point and β_1 the corresponding β cone, then we shall say that any point A_2 is *before* A_1 , provided that the two points are distinct and A_2 lies either on or inside the cone β_1 .

It is easy to see that :

(1) If a point A be *after* a point B , the point B is not *after* the point A , and is said to be *before* it.

(2) If A be any point, there is a point which is *after* A and also one which is *before* A .

(3) If a point A be *after* a point B , there is a point which is both *after* B and *before* A .

(4) If a point B be *after* a point A and a point C be *after* the point B , the point C is *after* the point A .

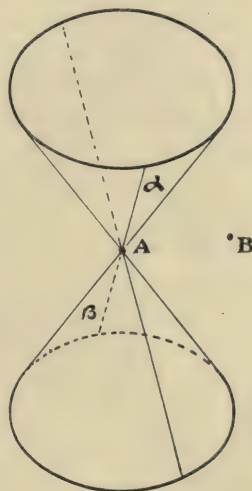


Fig. 1.

We cannot, however, say in this case that, if a point A be neither *before* nor *after* a point B , the points A and B must be identical.

We have here, in fact, that not only is A neither *before* nor *after* A , but also any element B lying in the region outside both the α and β cones of A is neither *before* nor *after* A . (See figure.)

Thus the fact that an element is neither *before* nor *after* an element is not sufficient in this case to enable us to say that the elements are identical.

It is to be observed, however, that if an element A be neither *before* nor *after* another element B , there are elements which are *after* both, and also elements which are *before* both; since the α cones of A and B intersect in this case, as do also the β cones of A and B .

We shall speak of the type of order thus obtained as *Conical Order*, but shall not confine ourselves in the use of the name to the geometrical

example considered above, or to the number of dimensions in which the example is presented.

The main point to be considered at present is that we may have two elements, of which one is neither *before* nor *after* the other, but which yet are not identical, without our being involved in any logical absurdity.

Let us now return to the consideration of elements of time.

We recognise *events* as taking place in time and as having a time order. We further recognise certain events as of an instantaneous character; such, for example, as two particles striking one another.

In speaking of events hereafter, unless otherwise stated, we shall refer to instantaneous events.

Two events may occur at the same instant; thus a particle *P* may strike a particle *R* at the same instant as another particle *Q* strikes it.

Events which occur at the same instant will be said to be *simultaneous*.

According to the view here put forward, the only events which are really simultaneous are events which occur at the same place.

The standpoint is practically this: that although the set of instants of which any one individual is directly conscious, or the set of instants which a single particle of matter occupies, are sets in linear order, yet the aggregate of all instants forms a set in *conical order*.

Thus, of any two events *A* and *B*, we may have *A before B* or *A after B*, or *A neither before nor after B*.

According to the view generally held, *A* being neither *before* nor *after B* is taken as equivalent to *A* and *B* being simultaneous. According to the view here adopted, this is only so when the events *A* and *B* occur at the same place.

If such events occur at different places, we are only entitled to say that the one is neither *before* nor *after* the other.

The standpoint is rendered fairly clear by a consideration of the geometrical type of conical order which we have explained.

If we consider, for instance, the case of a straight line which makes an angle with the axes of the cones not greater than their semi-vertical angle, it is evident that of any two distinct elements of such a line one must be *after* the other; and so, if an element *A* be neither *before* nor *after* an element *B*, the elements *A* and *B* must be identical.

Thus, although the whole aggregate of elements are in conical order, it is possible to select certain sets of elements of the aggregate

which have a linear order analogous to that possessed by the instants of time, of which any one individual is directly conscious.

Having thus given a brief outline of the idea of conical order as applied to instants of time, it is desirable to give some further justification for our adopting it.

Let us consider more closely what we practically mean when we say that one instant is *after* another, or one event *after* another.

One fact is clear :

If an instant B be distinct from an instant A , and if I, at the instant A , can produce any effect, however slight, at the instant B , then this is *sufficient* to imply that B is *after* A .

Now our contention is that we have here not merely a *sufficient* but also a *necessary* condition that B is *after* A .

Thus we have the following definition :

If an instant B be distinct from an instant A , then B will be said to be after A , if, and only if, it be abstractly possible for a person, at the instant A , to produce an effect at the instant B .

If this be granted, then it follows that if A and B be distinct instants, and if I, at the instant A , cannot produce any effect at the instant B , then B is *not after* A .

It does not, however, follow that B is *before* A , unless a person, at the instant B , can produce an effect at the instant A , since *before* and *after* are converse relations.

Thus the significance of an instant A being neither *before* nor *after* a distinct instant B is that I, at the instant A , should be unable to produce any effect at the instant B , and that another person, at the instant B , should be unable to produce any effect at the instant A .

The question arises : are there any grounds for thinking that instants can stand in such a relation ?

The answer appears to be in the affirmative.

Let us consider a flash of light or other electromagnetic disturbance to be sent out from a particle P at an instant A to a separate particle Q not in contact with P , and let the flash or disturbance be reflected back again directly to P .

Now, Fizeau's apparatus for determining what we usually speak of as the "Velocity of light" is an arrangement in which this is practically carried out, and it indicates that if the flash returns to P at an instant, say C , then C is *after* A .

If it be granted that the flash sent out from P arrives at Q at a

definite instant, say B , which is distinct from A and C , it follows from the meaning of *after* that B is *after* A and C is *after* B .

Now, there are strong reasons for thinking that no influence or material particle could be sent out from P at the instant A so as to arrive at Q at any instant *before* B ; and, similarly, there are strong reasons for thinking that no influence or material particle could be sent out from P at any instant *after* A so as to arrive at Q at the instant B .

If these be granted, we see that any instant at the particle P , which is *after* A , and *before* C , will, according to our view of the matter, be neither *before* nor *after* B .

Thus, if we consider Fizeau's arrangement, any instant at the sending apparatus, which is after the instant of departure of a flash of light and before the instant of its return, is neither before nor after the instant of arrival of the flash at the distant reflector.

As a matter of fact, we have no means of identifying any particular instant at the sending apparatus *after* the instant of departure and *before* the instant of return with the instant of arrival at the reflector.

Einstein attempts to identify the time of arrival with that midway between the times of departure and return, but we have already pointed out the logical difficulty in which this involves us. Further, Einstein, in order to determine the instant midway between the instants of departure and return, postulates the existence of a *clock*.

It does not appear a satisfactory mode of procedure to found a philosophical theory upon a complicated mechanism like a *clock* without any precise definition of what constitutes equal intervals of time.

According to the theory here put forward, we avoid both these difficulties and base the logical superstructure upon the ideas of *before* and *after*, giving to them the philosophical and physical meanings above described.

Thus, instead of starting from ordinary geometric cones with a definite angle and giving thereby an interpretation to *before* and *after*, it is proposed to reverse this process, and, starting from the ideas of *before* and *after*, to formulate in terms of them a system of postulates and definitions, and thereby build up a system of geometry.

Instead of speaking of α and β cones, we shall speak of α and β sub-sets, but shall find it often exceedingly suggestive to picture these to our minds by cones.

The physical interpretation is this:

If a flash of light or other instantaneous electromagnetic disturbance be sent out, let us say from a particle P at the instant A , so as to arrive directly at another particle Q at the instant B , then the instant B

lies in the α sub-set of the instant A , while the instant A lies in the β sub-set of the instant B .

The system of geometry thus built up will ultimately assume a sort of four dimensional character, or rather, we should say, any element of it is determined by four coordinates.

It thus appears that the theory of space becomes absorbed in the theory of time, spacial relations being regarded as the manifestation of the fact that the elements of time form a system in conical order: a conception which may be analysed in terms of the relations of after and before.

If I am directly conscious of an instant A , and if B be a distinct instant which is neither *before* nor *after* A , then B is an instant of which I am only indirectly aware, and so it assumes an *external character*.

We say that it is an instant *elsewhere*, and we can thus see in a general way how time relations and space relations can yet be both relations of one continuum.

We shall now proceed with the formal development of the theory briefly sketched above.

It will be observed that the postulates given generally consist of two parts marked (a) and (b) in which the relations of *after* and *before* are interchanged.

In some however, such as those numbered I, III and IV, the one part follows as a direct consequence of the mutual relations of *after* and *before*, while in others, such as V, these relations are involved symmetrically.

CONICAL ORDER

We shall suppose that we have a set of elements and that certain of these elements stand in a relation to certain other elements of the set which we denote by saying that one element is *after* another.

We shall further assume the following conditions:

POSTULATE I. If an element B be after an element A , then the element A is not after the element B .

Definition. If an element B be *after* an element A , then the element A will be said to be *before* the element B .

POSTULATE II. (a) If A be any element, there is at least one element which is after A .

(b) If A be any element there is at least one element which is before A .

POSTULATE III. If an element B be after an element A , and if an element C be after the element B , the element C is after the element A .

POSTULATE IV. If an element B be after an element A , there is at least one element which is both after A and before B .

POSTULATE V. If A be any element, there is at least one other element distinct from A which is neither before nor after A .

POSTULATE VI. (a) If A and B be two distinct elements, one of which is neither before nor after the other, there is at least one element which is after both A and B , but is not after any other element which is after both A and B .

(b) If A and B be two distinct elements, one of which is neither after nor before the other, there is at least one element which is before both A and B , but is not before any other element which is before both A and B .

Definition. (a) If A be any element of the set, then an element X will be said to be a member of the α sub-set of A provided X is either identical with A , or else provided there exists at least one element Y distinct from A and neither *before* nor *after* A and such that X is *after* both A and Y but is not *after* any other element which is *after* both A and Y .

(b) If A be any element of the set, then an element X will be said to be a member of the β sub-set of A provided X is either identical with A , or else provided there exists at least one element Y distinct from A and neither *after* nor *before* A and such that X is *before* both A and Y but is not *before* any other element which is *before* both A and Y .

If A be any element, then, by Post. V, there is at least one other element distinct from A which is neither *before* nor *after* A and so it follows directly by Post. VI (a) that there is at least one other element besides A which is a member of the α sub-set of A .

Similarly, by Post. VI (b), there is at least one other element besides A which is a member of the β sub-set of A .

Notation. We shall denote by α_1 and β_1 the sub-sets corresponding to an element A_1 , and by α_2 and β_2 those corresponding to an element A_2 , &c.

POSTULATE VII. (a) **If A_1 and A_2 be elements and if A_2 be a member of α_1 , then A_1 is a member of β_2 .**

(b) **If A_1 and A_2 be elements and if A_2 be a member of β_1 , then A_1 is a member of α_2 .**

POSTULATE VIII. (a) **If A_1 be any element and A_2 be any other element in α_1 , there is at least one other element distinct from A_2 which is a member both of α_1 and of α_2 .**

(b) **If A_1 be any element and A_2 be any other element in β_1 , there is at least one other element distinct from A_2 which is a member both of β_1 and of β_2 .**

THEOREM 1.

If A_1 be any element and A_2 be any other element in α_1 , then any element A_3 which is both after A_1 and before A_2 must be a member both of α_1 and β_2 .

By the definition of a member of the sub-set α_1 there exists at least one element, say A_4 , distinct from A_1 and neither *before* nor *after* A_1

and such that A_2 is *after* both A_1 and A_4 but is not *after* any other element which is *after* both A_1 and A_4 .

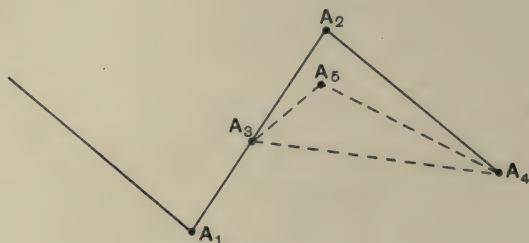


Fig. 2.

Then A_4 cannot be *after* A_3 , for if it were then, by Post. III, A_4 would be *after* A_1 contrary to hypothesis.

Further A_4 cannot be identical with A_3 , for then again we should have A_4 *after* A_1 contrary to hypothesis.

Again A_4 cannot be *before* A_3 , for then we should have A_2 *after* the element A_3 which would be *after* both A_1 and A_4 contrary to the hypothesis that A_2 is *after* both A_1 and A_4 but not *after* any other element which is *after* both A_1 and A_4 .

Thus A_4 is distinct from A_3 and is neither *before* nor *after* A_3 .

Now A_2 cannot be *after* any other element which is *after* both A_3 and A_4 , for if A_5 were such an element it would follow by Post. III that since A_3 is *after* A_1 we should have A_5 *after* A_1 .

Thus we should have A_2 *after* A_5 which would be *after* both A_1 and A_4 contrary to hypothesis.

Thus no such element as A_5 can exist and so A_2 satisfies the definition of being a member of α_3 .

Thus by Post. VII (a) it follows that A_3 is a member of β_2 .

Again by Post. VII (a) since A_2 is a member of α_1 it follows that A_1 is a member of β_2 , and so by a similar method we may prove that A_3 is a member of α_1 . Thus the theorem is proved.

We may state the results of this theorem as follows :

(a) If A_1 be any element and A_2 be any other distinct element in α_1 then A_1 is *before* A_2 , but is not *before* any other element *outside* the sub-set α_1 and *before* A_2 .

(b) If A_1 be any element and A_2 be any other distinct element in β_1 then A_1 is *after* A_2 , but is not *after* any other element *outside* the sub-set β_1 and *after* A_2 .

THEOREM 2.

(a) *If A_1 be any element and A_2 be any other element in α_1 , there is at least one other element in α_1 distinct from A_2 which is neither before nor after A_2 .*

Since A_2 is a member of α_1 it follows by Post. VII (a) that A_1 is a member of β_2 .

Thus there exists at least one other element, say A_3 , distinct from A_2 and neither *before* nor *after* A_2 and such that A_1 is *before* A_2 and A_3 , but is not *before* any other element which is *before* both A_2 and A_3 .

Thus A_1 satisfies the definition of being a member both of β_2 and β_3 and so, by Post. VII (b), A_3 is also a member of α_1 . Thus since A_3 is distinct from A_2 and neither *before* nor *after* A_2 , the theorem is proved.

(b) *If A_1 be any element and A_2 be any other element in β_1 , there is at least one other element in β_1 distinct from A_2 which is neither after nor before A_2 .*

Definition. If A_1 be any element and A_2 be any other element in α_1 , the *optical line* A_1A_2 is defined as the aggregate of all elements which lie either

- (1) both in α_1 and α_2 ,
- or (2) both in α_1 and β_2 ,
- or (3) both in β_1 and β_2 .

THEOREM 3.

(a) *If a be any optical line, there exists at least one element which is not an element of the optical line, but is before some element of it.*

If A_1 be any element and A_2 be any other element in α_1 then, by Post. VII (a), A_1 is a member of β_2 .

Thus by Theorem 2 (b) there is at least one other element in β_2 distinct from A_1 which is neither *after* nor *before* A_1 .

Call such an element A_3 .

Then since A_3 is in β_2 and distinct from A_2 it is *before* A_2 .

But A_3 cannot lie in the optical line A_1A_2 , for by the definition of the optical line A_1A_2 , in order to lie in it A_3 would require to lie also either in α_1 or β_1 .

But if A_3 lay in α_1 it would be either *after* A_1 or identical with A_1 , while if A_3 lay in β_1 it would be either *before* A_1 or identical with A_1 .

But A_3 is distinct from A_1 and is neither *after* nor *before* A_1 and therefore does not lie in the optical line A_1A_2 , although it is *before* A_2 an element of it.

(b) If a be any optical line, there exists at least one element which is not an element of the optical line, but is *after* some element of it.

POSTULATE IX. (a) If a be an optical line and if A_1 be any element which is not in the optical line but *before* some element of it, there is one single element which is an element both of the optical line a and the sub-set α_1 .

(b) If a be an optical line and if A_1 be any element which is not in the optical line but *after* some element of it, there is one single element which is an element both of the optical line a and the sub-set β_1 .

THEOREM 4.

(a) If A_1 be any element there is at least one other element which is *after* A_1 but is not a member of the sub-set α_1 .

Let A_2 be any other member of the sub-set α_1 distinct from A_1 .

Then A_2 is *after* A_1 and so by Post. IV there is at least one element, say A_3 , which is both *after* A_1 and *before* A_2 .

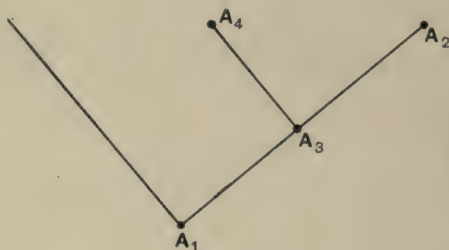


Fig. 3.

By Theorem 1 A_3 is a member both of α_1 and of β_2 and is therefore an element of the optical line A_1A_2 .

But since A_3 is a member of β_2 it follows that A_2 is a member of α_3 and so by Theorem 2 there is at least one other element in α_3 distinct from A_2 which is neither *before* nor *after* A_2 .

Let A_4 be such an element.

Then since A_4 is neither *before* nor *after* A_2 it cannot be a member either of β_2 or α_2 and so A_4 is not an element of the optical line A_1A_2 although it is *after* A_3 an element of it.

But since A_4 is a member of α_3 it follows by Post. VII (a) that A_3 is a member of the sub-set β_4 .

Thus A_3 is the *one single element* which by Post. IX (b) is an element both of the optical line and the sub-set β_4 .

But A_4 cannot be a member of α_1 , for then A_1 would be a member of β_4 and so A_1 would be a second element common to the optical line A_1A_2 and the sub-set β_4 , which is impossible by Post. IX (b).

Further A_4 is *after* A_3 and A_3 is *after* A_1 and therefore A_4 is after A_1 .

Thus A_4 is *after* A_1 but is not a member of the sub-set α_1 .

(b) *If A_1 be any element there is at least one other element which is before A_1 but is not a member of the sub-set β_1 .*

THEOREM 5.

If A_1 be any element and A_2 be any other element which is after A_1 , there is at least one other distinct element which is a member of both α_1 and β_2 .

Two cases arise: (1) A_2 may be a member of α_1 or (2) A_2 may not be a member of α_1 .

If A_2 is a member of α_1 then by Post. IV there is at least one element which is both *after* A_1 and *before* A_2 , and by Theorem 1 such an element is a member both of α_1 and β_2 .

Thus case (1) is proved.

Suppose next that A_2 is not a member of α_1 and let A_3 be any element of α_2 distinct from A_2 .

Then the optical line A_2A_3 , which for brevity we may call a , consists of the aggregate of all elements which lie either

(1) both in α_2 and α_3 ,

or (2) both in α_2 and β_3 ,

or (3) both in β_2 and β_3 .

Since A_2 is not a member of α_1 it follows that A_1 is not a member of β_2 and so, since A_1 is *before* A_2 it follows that A_1 is not an element of the optical line a .

Then by Post. IX (a) since A_1 is not an element of the optical line a but is *before* an element of it, it follows that there is one single element which is an element both of the optical line a and the sub-set α_1 .

Let A_4 be this element.

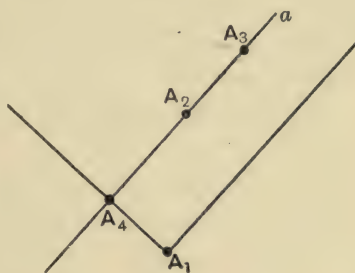


Fig. 4.

Then since we have supposed that A_2 is not a member of α_1 it follows that A_4 is not identical with A_2 .

Further A_4 cannot be *after* A_2 for then we should have A_2 *after* A_1 and *before* A_4 and so by Theorem 1 we should have A_2 a member of α_1 contrary to hypothesis.

Thus A_4 cannot be a member of α_2 and therefore since it is an element of the optical line a it must be a member of β_2 and β_3 .

Thus the element A_4 is a member of both α_1 and β_2 and so the theorem is proved.

THEOREM 6.

(a) *If A_1 be any element and A_2 be any other element in α_1 , while A_3 is an element distinct from A_2 , which is a member both of α_1 and of α_2 , then there is at least one other element which is a member of α_1 , of α_2 and of α_3 .*

By Post. VIII (a) since A_3 is an element of α_2 distinct from A_2 there is at least one other element distinct from A_3 which is a member both of α_2 and of α_3 . Call such an element A_4 . Then since A_4 is in α_3 and distinct from A_3 it is *after* A_3 .

Thus A_4 is *after* an element of the optical line A_1A_2 .

But A_4 is a member of α_2 and also of α_3 and so by Post. VII (a) A_2 and A_3 are each members of β_4 .

Now if A_4 were not in the optical line A_1A_2 it would follow by Post. IX (b) that there was *one single element* which was an element both of the optical line and the sub-set β_4 .

There are however *at least two elements* A_2 and A_3 with this property and so A_4 must be in the optical line A_1A_2 .

Also since A_4 is in α_2 it must also be in α_1 from the definition of the optical line.

Thus A_4 is a member of α_1 , of α_2 and of α_3 .

(b) *If A_1 be any element and A_2 be any other element in β_1 , while A_3 is an element distinct from A_2 , which is a member both of β_1 and of β_2 , then there is at least one other element which is a member of β_1 , of β_2 and of β_3 .*

THEOREM 7.

(a) *If X be any element of an optical line there is at least one element of the optical line which is after X .*

Let the optical line be defined by any element A_1 and another element A_2 in α_1 . Then X may lie either

- (1) both in α_1 and α_2 ,

or (2) both in α_1 and β_2 ,

or (3) both in β_1 and β_2 .

If X be not identical with A_2 then in cases (2) and (3) since X lies in β_2 , the element A_2 is *after* X .

If X be identical with A_2 then by Post. VIII (a) there is at least one other element distinct from A_2 which is a member both of α_1 and of α_2 and is therefore an element of the optical line.

Since such an element is not identical with A_2 it must be *after* A_2 ; that is to say it must be *after* X .

Next suppose X is in both α_1 and α_2 and is distinct from A_2 .

It follows by Theorem 6 (a) that there is at least one *other* element which is a member of α_1 and α_2 and of the α sub-set of X .

Since such an element is not identical with X and lies in the α sub-set of X it must be *after* X .

Further since it is an element both of α_1 and of α_2 it lies in the optical line. Thus in all cases there is at least one element of the optical line which is *after* X .

(b) *If X be any element of an optical line there is at least one element of the optical line which is before X .*

THEOREM 8.

(a) *If A_1 be any element and A_2 be any other element in α_1 , and if A_3 and A_4 be other distinct elements which are members of both α_1 and α_2 , one of the two elements A_3 and A_4 is in the α sub-set of the other.*

Since A_3 is in α_2 and distinct from A_2 therefore A_2 and A_3 define an optical line. Further since A_2 and A_3 both lie in α_1 therefore A_1 lies in both β_2 and β_3 .

Thus A_1 is an element of the optical line A_2A_3 .

But A_4 , since it is a member of α_1 and not identical with A_1 , is *after* A_1 .

That is to say, it is *after* an element of the optical line A_2A_3 .

If then A_4 were not an element of the optical line A_2A_3 there would, by Post. IX (b), be *one single element* which would be an element both of the optical line A_2A_3 and the sub-set β_4 .

But A_4 is a member both of α_1 and of α_2 and so both A_1 and A_2 are members of β_4 .

Thus since A_1 and A_2 are two distinct elements of the optical line A_2A_3 it follows that A_4 must be an element of the same optical line.

But A_4 is a member of α_2 and therefore by the definition of the optical line A_4 must be either a member of α_3 or of β_3 .

If A_4 be a member of β_3 then we should have A_3 a member of α_4 .

Thus one of the two elements A_3 and A_4 lies in the α sub-set of the other.

It also follows, since A_3 and A_4 are supposed to be distinct, that the one is *after* the other.

(b) *If A_1 be any element and A_2 be any other element in β_1 , and if A_3 and A_4 be other distinct elements which are members of both β_1 and β_2 , one of the two elements A_3 and A_4 is in the β sub-set of the other.*

It also follows, since A_3 and A_4 are supposed to be distinct, that the one is *before* the other.

THEOREM 9.

If a pair of elements be in an optical line defined by another pair of elements, then one of the first pair is in the α sub-set of the other.

Consider the optical line defined by the element A_1 and another element A_2 in α_1 . Suppose now in the first place that we have an element A_3 distinct from A_1 and A_2 and lying in the optical line.

Then by the definition of an optical line A_3 may be

- (1) both in α_1 and α_2 ,
- or (2) both in α_1 and β_2 ,
- or (3) both in β_1 and β_2 .

Thus if A_1 and A_3 be taken as a pair of elements in the optical line defined by A_1 and A_2 , we have in the first and second cases A_3 is in α_1 , while in the third we have A_3 in β_1 and consequently A_1 in α_3 . Thus one of the pair A_1, A_3 is in the α sub-set of the other.

Again if A_2 and A_3 be taken as a pair of elements in the optical line defined by A_1 and A_2 , we have in the first case A_3 is in α_2 , while in the second and third we have A_3 in β_2 and consequently A_2 in α_3 . Thus one of the pair A_2, A_3 is in the α sub-set of the other.

Next suppose that we have another element A_4 lying in the optical line and distinct from A_1, A_2 and A_3 .

Then there are the following possibilities:

$$A_3 \text{ both in } \alpha_1 \text{ and } \alpha_2 \text{ with } \begin{cases} A_4 \text{ both in } \alpha_1 \text{ and } \alpha_2 \dots\dots(1), \\ \text{or } A_4 \text{ both in } \alpha_1 \text{ and } \beta_2 \dots\dots(2), \\ \text{or } A_4 \text{ both in } \beta_1 \text{ and } \beta_2 \dots\dots(3). \end{cases}$$

$$\begin{aligned}
 A_3 \text{ both in } \alpha_1 \text{ and } \beta_2 \text{ with } & \begin{cases} A_4 \text{ both in } \alpha_1 \text{ and } \alpha_2 \dots\dots(4), \\ \text{or } A_4 \text{ both in } \alpha_1 \text{ and } \beta_2 \dots\dots(5), \\ \text{or } A_4 \text{ both in } \beta_1 \text{ and } \beta_2 \dots\dots(6). \end{cases} \\
 A_3 \text{ both in } \beta_1 \text{ and } \beta_2 \text{ with } & \begin{cases} A_4 \text{ both in } \alpha_1 \text{ and } \alpha_2 \dots\dots(7), \\ \text{or } A_4 \text{ both in } \alpha_1 \text{ and } \beta_2 \dots\dots(8), \\ \text{or } A_4 \text{ both in } \beta_1 \text{ and } \beta_2 \dots\dots(9). \end{cases}
 \end{aligned}$$

In case (1) by Theorem 8 (a) one of the two elements A_3 and A_4 is in the α sub-set of the other. Similarly in case (9) by Theorem 8 (b) one of the two elements A_3 and A_4 is in the β sub-set of the other, and therefore by Post. VII (b) one of them is in the α sub-set of the other.

Consider next case (2).

Since A_4 is in α_1 and distinct from A_1 it follows that A_4 is *after* A_1 .

Further since A_4 is in β_2 and distinct from A_2 we have A_2 *after* A_4 , and since A_3 is in α_2 and distinct from A_2 we have A_3 *after* A_2 .

Thus by Post. III, A_3 is *after* A_4 .

But since A_3 is in α_1 therefore A_1 is in β_3 , and so by Theorem 1 A_3 is *after* A_1 but is not *after* any element outside the sub-set β_3 which is *after* A_1 .

But we have shown that A_3 is *after* A_4 which is *after* A_1 , and therefore A_4 must lie in β_3 and consequently A_3 lies in α_4 .

Similarly in case (4) we may prove that A_4 must lie in α_3 .

In an analogous manner in case (8) since A_4 is in β_2 and distinct from A_2 we have A_4 is *before* A_2 .

Further, since A_4 is in α_1 and distinct from A_1 we have A_4 is *after* A_1 , and since A_3 is in β_1 and distinct from A_1 we have A_1 is *after* A_3 .

Thus by Post. III, A_4 is *after* A_3 or A_3 is *before* A_4 .

But since A_3 lies in β_2 therefore A_2 lies in α_3 , and so by Theorem 1 (a) A_3 is *before* A_2 but is not *before* any element outside the sub-set α_3 which is *before* A_2 .

But we have shown that A_3 is *before* A_4 which is *before* A_2 and therefore A_4 must lie in α_3 .

Similarly in case (6) we may prove that A_3 must lie in α_4 .

Consider next case (3).

We have A_4 in β_2 and therefore A_2 in α_4 .

Also we have A_2 in α_1 , and so A_1 in β_2 .

Further we have A_4 in β_1 , and so A_1 in α_4 .

Thus A_4 and A_2 determine an optical line which contains A_1 .

But A_3 is in α_2 , and being distinct from A_2 it must be *after* A_2 an element of the optical line determined by A_4 and A_2 .

Also since A_3 is in both α_1 and α_2 it follows that both A_1 and A_2 lie in β_3 .

But by Post. IX (b), if A_3 were not in the optical line determined by A_1 and A_2 there would be *one single element* which would be an element both of the optical line and the sub-set β_3 .

Thus since there are at least two distinct elements A_1 and A_2 common to the optical line and the sub-set β_3 it follows that A_3 must be an element of the optical line A_1A_2 . Further, since A_3 lies in α_2 it must, by the definition of the optical line, lie also in α_1 .

We may in a similar manner show in case (7) that A_4 must lie in α_3 .

We are thus left with only case (5) to prove.

Now since A_2 is an element distinct from A_1 and lying in α_1 , therefore, by Post. VIII (a), there is at least one other element distinct from A_2 which is a member both of α_1 and of α_2 .

Call such an element A_5 .

Now since A_3 is in β_2 and distinct from A_2 it follows that A_2 is *after* A_3 , and since A_5 is in α_2 and distinct from A_2 , therefore A_5 is *after* A_2 .

Thus by Post. III, A_5 is *after* A_3 .

Further, since A_5 is an element of α_1 , therefore A_1 is an element of β_5 and similarly A_2 is an element of β_5 .

But since A_1 is an element of β_5 it follows by Theorem 1 (b) that A_5 is *after* A_1 but is not *after* any element outside β_5 which is *after* A_1 .

But we have seen that A_5 is *after* A_3 , which, being an element of α_1 and not identical with A_1 , is *after* A_1 .

Thus A_3 must be an element of the sub-set β_5 .

Similarly A_4 must be an element of the sub-set β_5 .

Also both A_3 and A_4 are elements of β_2 and so by Theorem 8 (b) one of the two elements A_3 and A_4 is in the β sub-set of the other, and therefore by Post. VII (b) the one is in the α sub-set of the other.

Thus the theorem is true in all cases.

It follows directly from this theorem that of any two distinct elements in an optical line one is after the other.

THEOREM 10.

Any two elements of an optical line determine that optical line.

Let A_1 be any element and A_2 any other element in α_1 , then the optical line A_1A_2 is defined as the aggregate of all elements which lie either

- | | |
|----|---|
| | (1) both in α_1 and α_2 , |
| or | (2) both in α_1 and β_2 , |
| or | (3) both in β_1 and β_2 . |

Suppose A_3 and A_4 to be any pair of elements in the optical line A_1A_2 ; then by Theorem 9 one of the pair A_3, A_4 is in the α sub-set of the other.

We may suppose without loss of generality that it is A_4 which is in the sub-set α_3 .

Consider now any element A_5 of the optical line A_1A_2 such that A_5 is distinct from A_3 and A_4 .

Then by Theorem 9 there are the following possibilities :

A_4 in α_5 and also A_3 in α_5 (1),

A_4 in α_5 and also A_5 in α_3 (2),

A_5 in α_4 and also A_5 in α_3 (3),

A_5 in α_4 and also A_3 in α_5 (4).

Case (4) must however be excluded, for since A_3, A_4 and A_5 are supposed distinct we should have A_5 *after* A_4 and A_3 *after* A_5 and therefore, by Post. III, A_3 *after* A_4 .

We however supposed A_4 to be *after* A_3 and by Post. I we cannot have also A_3 *after* A_4 . Thus case (4) is impossible.

The three permissible cases may be expressed thus :

A_5 both in β_3 and β_4 (1),

A_5 both in α_3 and β_4 (2),

A_5 both in α_3 and α_4 (3).

Thus in all cases A_5 lies in the optical line defined by A_3 and A_4 .

Similarly it may be shown that every element in the optical line defined by A_3 and A_4 lies in the optical line defined by A_1 and A_2 .

Thus the optical lines A_1A_2 and A_3A_4 are identical.

THEOREM 11.

If A_3 and A_4 be any two elements of an optical line A_1A_2 there is at least one element of the optical line which is after the one and before the other.

Since A_3 and A_4 are both elements of the same optical line the one must be in the α sub-set of the other by Theorem 9.

We shall suppose that A_4 lies in α_3 .

Then since A_3 and A_4 are distinct, A_4 will be *after* A_3 , and so by Theorem 5 there is at least one other distinct element which is a member both of α_3 and of β_4 .

Call such an element A_5 .

Then A_5 is in the optical line A_3A_4 , and therefore by Theorem 10 in the optical line A_1A_2 .

Further since A_5 is distinct from A_3 and A_4 it must be *after* A_3 and *before* A_4 .

From the preceding results it follows that an optical line contains an infinite number of elements.

THEOREM 12.

If an element A_1 be before an element of an optical line a , and be also after an element of a , then A_1 must be itself an element of the optical line a .

Suppose that A_1 is *before* the element A_2 of a and also *after* the element A_3 of a .

Then by Post. I A_3 cannot be identical with A_2 , and by Theorem 9 one of the elements A_2 and A_3 must be in the α sub-set of the other.

Since A_1 is *after* A_3 and A_2 is *after* A_1 it follows that A_2 is *after* A_3 and so it must be A_2 which is in the α sub-set of A_3 .

But by Theorem 1 it follows that A_1 must lie in α_3 and also in β_2 , and accordingly A_1 lies in the optical line A_3A_2 .

Thus since, by Theorem 10, any two elements of an optical line determine that optical line, it follows that A_1 lies in the optical line a .

THEOREM 13.

(a) *If A_1 be any element and A_2 be any other element in α_1 and if A_3 be any element in α_1 which is either before or after A_2 , then A_3 lies in the optical line A_1A_2 .*

(1) Suppose A_3 is *before* A_2 .

Then since A_3 lies in α_1 it must be either identical with A_1 : in which case it lies in the optical line A_1A_2 ; or else A_3 is *after* A_1 : in which case by Theorem 1 A_3 must lie both in α_1 and β_2 and therefore must lie in the optical line A_1A_2 .

(2) Suppose A_3 is *after* A_2 .

Then A_3 lies in α_1 and A_2 is *after* A_1 and *before* A_3 and therefore, by Theorem 1, A_2 must lie both in α_1 and β_3 .

But if A_2 lies in β_3 , it follows by Post. VII (b) that A_3 lies in α_2 .

Thus A_3 lies both in α_1 and α_2 and therefore lies in the optical line A_1A_2 .

(b) *If A_1 be any element and A_2 be any other element in β_1 and if A_3 be any element in β_1 which is either after or before A_2 , then A_3 lies in the optical line A_1A_2 .*

THEOREM 14.

(a) *If A_1 be any element and A_2 be any other element in α_1 and A_3 be any element in α_1 distinct from A_2 which is neither before nor after A_2 , then A_3 is neither before nor after any element of the optical line A_1A_2 which is after A_1 .*

The element A_3 cannot lie in the optical line A_1A_2 , for then since it is distinct from A_2 it would be either *before* or *after* it, contrary to hypothesis.

Now any element of the optical line A_1A_2 which is *after* A_1 must lie in α_1 .

Let A_4 be any such element.

Then if A_3 were either *before* or *after* A_4 it would by Theorem 13 lie in the optical line A_1A_4 , which by Theorem 10 is identical with the optical line A_1A_2 , and this we have shown to be impossible.

Thus A_3 cannot be either *before* or *after* any element of the optical line A_1A_2 which is *after* A_1 .

(b) *If A_1 be any element and A_2 be any other element in β_1 and A_3 be any element in β_1 distinct from A_2 which is neither after nor before A_2 , then A_3 is neither after nor before any element of the optical line A_2A_1 which is before A_1 .*

POSTULATE X. (a) **If a be an optical line and if A be any element not in the optical line but before some element of it, there is one single optical line containing A and such that each element of it is before an element of a .**

(b) **If a be an optical line and if A be any element not in the optical line but after some element of it, there is one single optical line containing A and such that each element of it is after an element of a .**

THEOREM 15.

(a) *If each element of one optical line be before an element of another optical line the two optical lines cannot have an element in common.*

It is evident from Theorem 10 that two distinct optical lines cannot have more than one element in common.

Consider now any two optical lines a and b having the element A_1 in common and let A_2 be another element of a in α_1 .

Then by Theorem 13 if A_3 be any other distinct element in α_1 which is either *before* or *after* A_2 , then A_3 lies in the optical line A_1A_2 : that is in a .

Further if A_3 be *after* A_1 it cannot lie in b , for then the optical lines a and b would have the two elements A_1 and A_3 in common and therefore would be identical.

Thus A_2 cannot be *before* any element of b which is *after* A_1 .

But A_2 is *after* A_1 and is therefore *after* any element of b which is *before* A_1 .

Thus by Post. I A_2 cannot be *before* A_1 or any element of b which is *before* A_1 .

It follows that A_2 is not *before* any element of the optical line b and therefore, if two optical lines a and b have an element in common, we cannot have each element of a *before* an element of b .

Thus conversely if each element of an optical line a be *before* an element of another optical line b the two optical lines cannot have an element in common.

(b) *If each element of one optical line be after an element of another optical line the two optical lines cannot have an element in common.*

THEOREM 16.

(a) *If each element of an optical line a be before an element of another optical line b , then through each element of a there is one single optical line which contains also an element of b .*

By Theorem 15 an element of a cannot also be an element of b .

Suppose then that A_1 be any element of a .

Then A_1 is not an element of b , but is *before* an element of b and therefore by Post. IX (a) there is *one single element*, say A_2 , which is an element both of the optical line b and the sub-set α_1 . Since A_2 cannot be identical with A_1 , it follows that A_1 and A_2 determine an optical line which contains an element of a and also an element of b .

Further there cannot be more than one optical line through A_1 which contains also an element of b ; for such an element of b must, by Theorem 9, lie either in α_1 or β_1 .

But by Post. IX (a) there is only *one single element* common to b and the sub-set α_1 , and so if such an element of b existed it would have to lie in β_1 .

Call such a hypothetical element A_3 .

Then since A_3 is supposed to lie in β_1 , we should have A_1 in α_3 .

But A_2 lies in α_1 and so A_1 lies in β_2 , and thus if such an element as A_3 existed, A_1 would lie in the optical line A_3A_2 : that is, in the optical line b , which is impossible, and so there cannot be any such element as A_3 .

Thus there is only one single optical line through A_1 which contains also an element of b .

(b) *If each element of an optical line a be after an element of another optical line b , then through each element of a there is one single optical line which contains also an element of b .*

Definition. If two distinct optical lines have an element in common they will be said to *intersect* one another in that element.

If A_1 and A_2 be two distinct elements one of which is neither *before* nor *after* the other, then we know by Post. VI that there is at least one element, say X , which is *after* both A_1 and A_2 , but is not *after* any other element which is *after* both A_1 and A_2 .

From the definition of α sub-sets it follows that X lies both in α_1 and α_2 , so that there is *at least one element* which is a member both of α_1 and α_2 . Similarly there is *at least one element* which is a member both of β_1 and β_2 .

These remarks prepare the way for Postulate XI (a) and (b).

POSTULATE XI. (a) If A_1 and A_2 be two distinct elements one of which is neither before nor after the other and X be an element which is a member both of α_1 and α_2 , then there is at least one other element distinct from X which is a member both of α_1 and α_2 .

(b) If A_1 and A_2 be two distinct elements one of which is neither after nor before the other and X be an element which is a member both of β_1 and β_2 , then there is at least one other element distinct from X which is a member both of β_1 and β_2 .

The above is the first of our postulates which requires more than two dimensions for its representation.

It is to be noted that it may easily be combined with Postulate VI as follows:

(a) *If A and B be two distinct elements one of which is neither before nor after the other, there are at least two distinct elements either of which is after both A and B but is not after any other element which is after both A and B .*

(b) *If A and B be two distinct elements one of which is neither after nor before the other, there are at least two distinct elements either of which is before both A and B but is not before any other element which is before both A and B .*

THEOREM 17.

(a) If A_1 and A_2 be any two distinct elements, one of which is neither before nor after the other, and if A_3 and A_4 be distinct elements which lie both in α_1 and α_2 , then one of these latter two elements is neither before nor after the other.

By the definition of α sub-sets A_3 is *after* both A_1 and A_2 but is not *after* any other element which is *after* both A_1 and A_2 .

Similarly A_4 is *after* both A_1 and A_2 but is not *after* any other element which is *after* both A_1 and A_2 .

Thus A_3 is not *after* A_4 , and A_4 is not *after* A_3 .

Thus A_3 is neither *before* nor *after* A_4 .

(b) If A_1 and A_2 be any two distinct elements, one of which is neither after nor before the other, and if A_3 and A_4 be distinct elements which lie both in β_1 and β_2 , then one of these latter two elements is neither after nor before the other.

THEOREM 18.

(a) If A_1 be any element and A_2 and A_3 be two other distinct elements of α_1 , one of which is neither before nor after the other, there is at least one other distinct element in α_1 which is neither before nor after A_2 and neither before nor after A_3 .

Since A_2 is a member of α_1 , therefore A_1 is a member of β_2 .

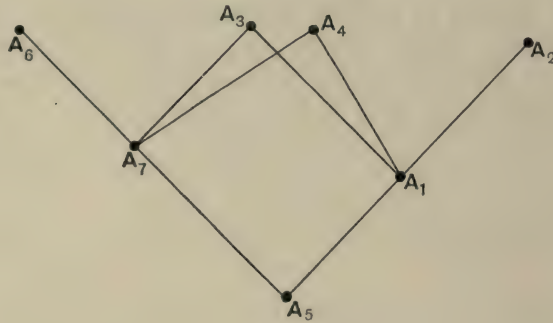


Fig. 5.

Thus by Post. VIII (b) there is at least one other element distinct from A_1 which is a member both of β_2 and of β_1 .

Call such an element A_5 .

Then A_1 and A_2 are both members of α_5 .

Thus by Theorem 2 (a) there is at least one other element in α_5 distinct from A_1 which is neither *before* nor *after* A_1 .

Call such an element A_6 .

Now A_3 cannot lie in α_5 for then, as it is an element of α_1 , it would lie in the optical line A_5A_1 along with A_2 and so A_2 and A_3 would either be identical or else A_2 would be either *before* or *after* A_3 , contrary to hypothesis.

Now A_3 is *after* A_1 and A_1 is *after* A_5 and so by Post. III A_3 is *after* A_5 , and since A_3 is not an element of α_5 it cannot lie in the optical line A_5A_6 .

Thus by Post. IX (b) there is *one single element* (say A_7) which is an element both of the optical line A_5A_6 and the sub-set β_3 .

Now A_5 cannot be *after* A_7 , for A_7 lies in β_3 , and so by Theorem 1 (b) A_3 is *after* A_7 but is not *after* any element outside the sub-set β_3 which is *after* A_7 .

But A_3 is *after* A_5 , and since A_3 does not lie in α_5 , therefore A_5 does not lie in β_3 .

Thus A_5 is not *after* A_7 .

Also A_5 cannot coincide with A_7 for then it would be in β_3 .

Thus A_7 must be *after* A_5 , and so by Theorem 14 A_1 is neither *before* nor *after* A_7 .

Now A_3 lies both in α_1 and in α_7 , and so by Post. XI (a) there is at least one other distinct element, say A_4 , which lies both in α_1 and in α_7 .

Then by Theorem 17 A_4 is neither *before* nor *after* A_3 .

Further A_4 cannot be either *before* or *after* A_2 , for since A_2 and A_4 are both members of α_1 it would follow by Theorem 13 that A_4 lay in the optical line A_1A_2 .

This would also be the case if A_4 coincided with A_2 .

But then (since A_4 is *after* A_1 and therefore *after* A_5) we should have A_4 in α_5 and A_1 and A_7 both in α_5 and β_4 , and thus A_1 and A_7 would lie in one optical line.

Thus A_1 and A_7 would either coincide or else the one would be *after* the other, which is impossible.

Thus A_4 is neither *before* nor *after* A_2 and is neither *before* nor *after* A_3 and is distinct from either.

(b) If A_1 be any element and A_2 and A_3 be two other distinct elements of β_1 , one of which is neither *after* nor *before* the other, there is at least one other distinct element in β_1 which is neither *after* nor *before* A_2 and neither *after* nor *before* A_3 .

THEOREM 19.

If A_1 be any element there are at least three distinct optical lines containing A_1 .

Let A_2 be any element in α_1 distinct from A_1 .

Then by Theorem 2 (a) there is at least one other element in α_1 distinct from A_2 which is neither *before* nor *after* A_2 .

Call such an element A_3 .

Further by Theorem 18 there is at least one other distinct element in α_1 which is neither *before* nor *after* A_2 and neither *before* nor *after* A_3 .

Call such an element A_4 .

Then A_1 and A_2 determine one optical line; A_1 and A_3 determine a second optical line; A_1 and A_4 determine a third optical line.

These are all distinct and all contain A_1 .

If a be an optical line and if A be any element not in the optical line but *before* some element of it we have by Post. X (a) one single optical line containing A and such that each element of it is *before* an element of A .

Further we have seen in Theorem 16 that there is one single optical line containing A and also intersecting a .

Also by Theorem 19 there are at least three optical lines containing A and so there must be at least one optical line containing A in addition to the two particular ones which we have already mentioned.

Similarly if a be an optical line and if A be any element not in the optical line but *after* some element of it, there is one single optical line containing A and such that each element of it is *after* an element of a and there is one single optical line containing A and intersecting a .

In addition to these two particular optical lines Theorem 19 shows that there is at least one other optical line containing A .

These considerations prepare the way for Postulate XII (a) and (b).

POSTULATE XII. (a) If a be an optical line and if A be any element not in the optical line but before some element of it, then each optical line through A , except the one which intersects a and the one of which each element is before an element of a , has one single element which is neither before nor after any element of a .

(b) If a be an optical line and if A be any element not in the optical line but after some element of it, then each optical line through A , except the one which intersects a and the one of which each element is after an element of a , has one single element which is neither after nor before any element of a .

THEOREM 20.

(a) *If each element of an optical line a be after an element of a distinct optical line b , then each element of b is before an element of a .*

Let A_1 be any element of a ; then since A_1 is not in b but *after* an element of b , there is one single element (say A_2) common to the optical line b and the sub-set β_1 (Post. IX (b)).

Then A_2 is not an element of a but is *before* the element A_1 of a and so by Post. X (a) there is *one single optical line* (say c) containing A_2 and such that each element of it is *before* an element of a .

Now b cannot be identical with the optical line A_2A_1 , for then a and b would have the element A_1 in common, which is impossible by Theorem 15 (b).

Suppose now, if possible, that b is not identical with c ; then by Post. XII (a) there will be *one single element* in b (say A_3) which will be neither *before* nor *after* any element of a .

Now A_3 cannot be *before* A_2 , for if so we should have A_2 *after* A_3 and A_1 *after* A_2 and therefore by Post. III A_1 *after* A_3 , contrary to the hypothesis that A_3 is neither *before* nor *after* any element of a .

Also since A_2 is *before* A_1 we cannot have A_3 identical with A_2 .

Suppose now that A_3 is *after* A_2 and consider an element A_4 in b and *after* A_3 .

Since there can only be one element in b which is neither *before* nor *after* any element of a , it follows that A_4 must be either *before* or *after* some element of a .

Since A_3 is *before* A_4 it would follow, if A_4 were *before* an element of a , that A_3 was also *before* an element of a , contrary to hypothesis.

We must therefore suppose that A_4 is *after* an element of a .

Then since A_4 cannot be an element of a it would follow by Post. IX (b) that there was *one single element* which was an element both of the optical line a and the sub-set β_4 .

Call such an element A_5 .

Now since A_5 is an element of a it must be *after* some element of b , and since by Theorem 15 (b) A_5 cannot be an element of b , there must by Post. IX (b) be one single element (say A_6) which is an element both of the optical line b and the sub-set β_5 .

Then A_5 must be in α_6 .

But we were led to the conclusion that A_5 lay in β_4 and since A_5 could not be both *before* and *after* the same element, A_6 would have to be distinct from A_4 and both A_6 and A_4 are supposed to be elements of the optical line b .

Thus A_5 would have to lie in the optical line b , which is impossible by Theorem 15 (b).

Thus the supposition that b is distinct from c leads to a contradiction and therefore is not true.

Thus b must be identical with c and so each element of b is *before* an element of a .

(b) *If each element of an optical line a be before an element of a distinct optical line b , then each element of b is after an element of a .*

THEOREM 21.

If a be an optical line and if A_1 be any element which is neither before nor after any element of a , there is one single optical line containing A_1 and such that no element of it is either before or after any element of a .

Let A_2 be any selected element of a ; then A_1 is neither *before* nor *after* A_2 , and so by Post. VI (b) an element exists which is a member both of β_1 and of β_2 .

Call such an element A_3 .

Now A_3 is *before* A_2 , an element of a , and does not lie in a and therefore by Post. X (a) there is *one single optical line* (say c) containing A_3 and such that each element of c is *before* an element of a .

Further A_1 is *after* A_3 but is not *before* any element of a and so does not lie in c .

Thus by Post. X (b) there is *one single optical line* (say b) containing A_1 and such that each element of b is *after* an element of c .

Consider now any element A_4 other than A_1 in the optical line b ; then A_4 cannot be an element of a since otherwise A_1 would be either *before* or *after* an element of a , contrary to hypothesis.

Suppose now if possible that A_4 is *after* some element of a .

Then by Post. X (b) there is *one single optical line* (say d) containing A_4 and such that each element of d is *after* an element of a .

But since each element of a is *after* an element of c therefore by Post. III each element of d is *after* an element of c .

But by Post. X (b) there is only *one single optical line* containing A_4 which has this property and the optical line b is such a one.

Thus the optical line d must be identical with the optical line b .

Thus each element of b would be *after* an element of a , contrary to the hypothesis that A_1 was neither *before* nor *after* any element of a .

Thus A_4 is not *after* any element of a .

Next suppose if possible that A_4 is *before* some element (say A_5) of a .

Then A_5 is not an element of b , but is *after* an element of b , and so by Post. X (b) there is *one single optical line* (say e) containing A_5 and such that each element of e is *after* an element of b .

But each element of b is *after* an element of c and so by Post. III each element of e is *after* an element of c .

There is however by Post. X (b) only *one single optical line* containing A_5 and having this property, and a is such an optical line.

Thus e must be identical with a and so each element of a must be *after* an element of b .

But if this were so then by Theorem 20 (a) each element of b must be *before* an element of a , contrary to the hypothesis that A_1 is neither *before* nor *after* any element of a .

Thus A_4 is not *before* any element of a , and so no element of b is either *before* or *after* any element of a .

We have thus shown that there is one optical line containing A_1 and having this property.

We have now to show that there is only one.

Consider any optical line containing A_1 other than the optical lines b and A_3A_1 .

Call such an optical line f .

Then by Post. XII (b) there is *one single element* in f (say A_6) such that A_6 is neither *before* nor *after* any element in c .

If then we take any element A_7 in f and *before* A_6 , such an element cannot be *after* any element in c , for then A_6 being *after* A_7 would be *after* an element of c , contrary to hypothesis.

Also since there is only one element having the property of A_6 and lying in f , therefore A_7 must be *before* some element of c .

But this element is *before* some element of a and so A_7 is *before* some element of a .

Thus there is only one optical line containing A_1 and such that no element of it is either *before* or *after* any element of a .

THEOREM 22.

If a be an optical line and A_1 be any element which is neither before nor after any element of a while b is the one single optical line containing A_1 and such that no element of it is either before or after any element of a , then every optical line through A_1 , with the exception of b , is divided by A_1 into elements which are before an element of a and elements which are after an element of a .

We proved in Theorem 21 that there is only one optical line through A_1 having the property of b .

Thus if we take any other optical line d through A_1 there must be at least one element of d which is either *before* or *after* some element of a .

Suppose first that there is an element A_3 which is *before* some element of a .

Then A_3 cannot be *after* A_1 , for since there is an element of a *after* A_3 there would by Post. III be an element of a *after* A_1 , contrary to hypothesis.

Thus A_3 must be *before* A_1 .

Further A_3 cannot be an element of a , for then A_1 would be *after* an element of a , contrary to hypothesis.

Thus A_3 is not an element of a but *before* an element of it, and so by Post. IX (a) there is *one single element* (say A_2) which is an element both of the optical line a and the sub-set α_3 .

Further by Post. X (a) there is *one single optical line* (say c) containing A_3 and such that each element of it is *before* an element of a .

Then by Post. XII (a) since the optical line d contains A_3 and is not identical with either of the optical lines A_3A_2 or c it follows that there is *one single element* of d which is neither *before* nor *after* any element of a .

But by hypothesis A_1 has this property and so every other element of d is either *before* or *after* an element of a .

However, as we have already seen, an element which is *after* A_1 in d cannot be *before* an element of a and so it must be *after* an element of a .

Similarly an element which is *before* A_1 in d cannot be *after* an element of a for then A_1 would be *after* an element of a contrary to hypothesis, and so an element which is *before* A_1 in d must be *before* an element of a .

We arrive at the same conclusion if we start off by supposing the existence in d of an element A_3' which is *after* some element of a . Thus the theorem is proved.

THEOREM 23.

(a) *If each element of each of two distinct optical lines a and b be after elements of a third optical line c , and if one element A_1 of the optical line b be after some element of the optical line a , then each element of b is after an element of a .*

Let b' be the *one single optical line* containing A_1 and such that each element of b' is *after* an element of a .

Then since each element of a is *after* an element of c therefore by Post. III each element of b' is *after* an element of c .

But by hypothesis each element of b is *after* an element of c , and b contains A_1 an element not in the optical line c but *after* some element of it.

Thus by Post. X (b), since there is only one single optical line containing A_1 and having this property, it follows that b' must be identical with b .

Thus each element of b is *after* an element of a .

(b) *If each element of each of two distinct optical lines a and b be before elements of a third optical line c , and if one element A_1 of the optical line b be before some element of the optical line a , then each element of b is before an element of a .*

THEOREM 24.

(a) *If each element of each of two distinct optical lines a and b be after elements of a third optical line c , and if one element A_1 of the optical line b be neither before nor after any element of the optical line a , then no element of the optical line b is either before or after any element of the optical line a .*

Since A_1 is not an element of c but is *after* some element of it, therefore by Post. IX (b) there is one single element (say A_3) which is common to the optical line c and the sub-set β_1 .

Then since A_3 is not an element of a , but is *before* an element of a (Theorem 20 (a)), therefore by Post. IX (a) there is one single element (say A_2) which is common to the optical line a and the sub-set α_3 .

The demonstration then follows as in Theorem 21.

(b) *If each element of each of two distinct optical lines a and b be before elements of a third optical line c , and if one element A_1 of the optical line b be neither after nor before any element of the optical line a , then no element of the optical line b is either after or before any element of the optical line a .*

This may be demonstrated in an analogous manner.

THEOREM 25.

(a) *If an optical line a be such that no element of it is either before or after any element of the optical line c , and if another optical line b be such that each element of it is before an element of c , then each element of b is before an element of a .*

Since each element of b is *before* an element of c , it follows by Theorem 20 (b) that each element of c is *after* an element of b .

Let A_1 be any element of c .

Then since A_1 is not an element of b but is *after* an element of b , there is one single element common to the optical line b and the sub-set β_1 (Post. IX (b)).

Let A_2 be this element.

Then A_2 and A_1 determine an optical line.

But by Theorem 22 every optical line containing A_1 except c is divided by A_1 into elements which are *before* an element of a and elements which are *after* an element of a , and since A_2 is *before* A_1 and lies in the optical line A_1A_2 , it follows that A_2 is also *before* an element of a and is not an element of a .

Thus by Post. IX (a) there is one single element (say A_3) common to the optical line a and the sub-set α_3 .

Now A_3 is neither *before* nor *after* any element of c and therefore if an optical line a' be taken through A_3 such that each element of it is *after* an element of b , then by Theorem 24 (a) no element of a' is either *before* or *after* any element of c .

But by Theorem 21 there is only one optical line through A_3 having this property and a is such an optical line.

Thus a' is identical with a and so each element of a is *after* an element of b and thus by Theorem 20 (a) each element of b is *before* an element of a .

(b) *If an optical line a be such that no element of it is either after or before any element of the optical line c , and if another optical line b be such that each element of it is after an element of c , then each element of b is after an element of a .*

THEOREM 26.

(a) *If each element of an optical line a be after an element of a distinct optical line c , and each element of another optical line b be before an element of c , then each element of a is after an element of b .*

By Theorem 20 (b) each element of c is *after* an element of b and since each element of a is *after* an element of c , therefore by Post. III each element of a is *after* an element of b .

(b) *If each element of an optical line a be before an element of a distinct optical line c , and each element of another optical line b be after an element of c , then each element of a is before an element of b .*

THEOREM 27.

If two distinct optical lines a and b be such that no element of either of them is either before or after any element of a third optical line c , then no element of a is either before or after any element of b .

For suppose, if possible, that some element A_1 of a is *after* an element of b ; then A_1 cannot lie in b and by Post. IX (b) there is one single element (say A_2) common to the optical line b and the sub-set β_1 .

But by Theorem 22 every optical line through A_1 except a is divided by A_1 into elements which are *before* an element of c and elements which are *after* an element of c .

Thus since A_2 and A_1 determine an optical line through A_1 , and since A_2 is *before* A_1 , therefore A_2 must be *before* an element of c , contrary to the hypothesis that no element of b is either *before* or *after* any element of c .

Similarly if we suppose A_1 to be *before* an element of b we are led to a conclusion contrary to hypothesis.

Thus no element of a is either *before* or *after* any element of b .

Definitions. An optical line a will be said to be parallel to a second distinct optical line b when either:

- (1) each element of a is *after* an element of b ,
- or (2) each element of a is *before* an element of b ,
- or (3) no element of a is either *before* or *after* any element of b .

In case (1) a will be said to be an *after-parallel* of b .

In case (2) a will be said to be a *before-parallel* of b .

In case (3) a will be said to be a *neutral-parallel* of b .

It follows from these definitions in conjunction with Theorem 20 that:

If an optical line a be parallel to an optical line b , then the optical line b is parallel to the optical line a .

Again if a be any optical line and A be any element *not in the optical line*, A may be *before* an element of a , or may be *after* an element of a , but by Theorem 12 A cannot be *before* one element of a and *after* another element of a .

By Post. XII (a) and (b) it follows that A may be neither *before* nor *after* any element of a .

If A be *before* an element of a , then by Post. X (a) there is one single parallel to a containing A .

If A be *after* an element of a , then by Post. X (b) there is one single parallel to a containing A .

If A be neither *before* nor *after* any element of a , then by Theorem 21 there is one single parallel to a containing A .

Thus we can say in general:

If a be any optical line and A be any element which is not in the optical line, then there is one single optical line parallel to a and containing A .

Further, combining Theorems 23 (a), 23 (b), 24 (a), 24 (b), 25 (a), 25 (b), 26 (a), 26 (b), 27, we have the general result that:

If two distinct optical lines a and b are each parallel to a third optical line c , then the optical lines a and b are parallel one to another.

Definition. If a and b be any pair of distinct optical lines one of which is an after-parallel of the other, then the aggregate of all elements of all optical lines which intersect both a and b will be called an *acceleration plane**.

THEOREM 28.

If a be an optical line there are an infinite number of distinct acceleration planes which all contain a .

From Post. XII (a) and (b) it follows that there is at least one element, say A_1 , which is neither *before* nor *after* any element of a .

If b be the one optical line through A_1 such that no element of it is either *before* or *after* any element of a , then by Theorem 22 every optical line through A_1 except b is divided by A_1 into elements which are *before* an element of a and elements which are *after* an element of a .

Let f be one particular optical line containing A_1 and distinct from b .

Let A_2 be any element in f other than A_1 ; then A_2 must be either *before* or *after* some element of a but is not itself an element of a .

Thus if an optical line c be taken through A_2 parallel to a then c is either a *before* or *after*-parallel of a and therefore along with a serves to define an acceleration plane.

Let A_3 be another element of f distinct from A_2 .

Then in order that A_3 should lie in the acceleration plane defined by a and c it would have to lie in an optical line intersecting both a and c .

* The reason for adopting this name is that, as will be seen hereafter, any acceleration of a particle determines an acceleration plane.

But since A_3 is distinct from A_2 and lies in the optical line f which also contains A_2 it must be either *before* or *after* A_2 , and so by Post. IX (a) or Post. IX (b) there must be *one single element* which is an element both of the optical line c and the sub-set α_3 or β_3 as the case may be.

But the element A_2 is such an element and therefore the optical line f containing A_3 and A_2 is the only optical line which intersects c and contains A_3 .

Thus in order that A_3 should lie in the acceleration plane defined by a and c it would be necessary for f to intersect a and this we know it does not do since if it did A_1 would be either *before* or *after* an element of a , contrary to hypothesis.

If then A_3 be distinct from A_1 it is either *before* or *after* an element of a and so if we take the optical line through A_3 parallel to a , it will be either a *before* or *after-parallel* of a .

Call such an optical line d .

Then d and a define another acceleration plane which is distinct from that defined by c and a , since the latter does not contain A_3 .

If any other element A_n in the optical line f be selected other than A_2 or A_3 and an optical line be taken through it parallel to a , then provided A_n is distinct from A_1 , the parallel to a through A_n will, along with a , define an acceleration plane distinct from the others.

Thus each element of f except A_1 corresponds to a distinct acceleration plane and the number of elements in f is infinite, while all the acceleration planes contain a .

Thus there are an infinite number of distinct acceleration planes all containing the optical line a .

From the last theorem it follows directly that it is permissible to speak of three or more acceleration planes which have two elements in common.

This prepares the way for Postulate XIII.

POSTULATE XIII. If two distinct acceleration planes have two elements in common, then any other acceleration plane containing these two elements contains all elements common to the two first-mentioned acceleration planes.

THEOREM 29.

If a and b be two distinct optical lines and if a be an after-parallel of b , then if c and d be two other distinct optical lines intersecting both a and b , one of these latter two optical lines is an after-parallel of the other.

Let the optical line c intersect b in A_1 and a in A_2 and let the other optical line d intersect b in A_3 and a in A_4 .

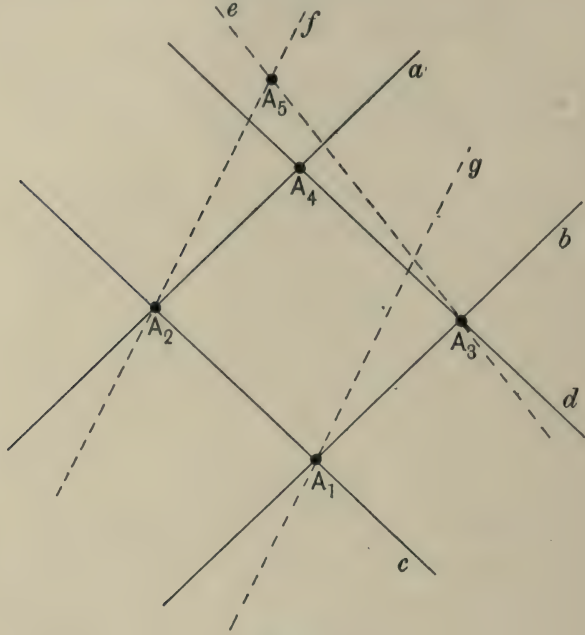


Fig. 6.

Then, by Theorem 16 (a), it is not possible for A_1 and A_3 to be coincident while A_2 and A_4 are distinct; while, by Theorem 16 (b), it is not possible for A_2 and A_4 to be coincident while A_1 and A_3 are distinct.

We may suppose without loss of generality that A_3 is *after* A_1 .

Then since a is an after-parallel of b we must have A_4 *after* A_3 and therefore by Post. III A_4 is *after* A_1 , or A_1 *before* A_4 .

Further, since a is an after-parallel of b , and since A_1 and A_2 lie in the optical line c , we must have A_2 *after* A_1 and therefore A_2 must lie in α_1 .

Thus by Theorem 1 (a) A_1 is *before* A_2 but is not *before* any element outside the sub-set α_1 which is *before* A_2 and so, since A_1 is *before* A_4 , and there is only one element common to the optical line a and the sub-set α_1 , it follows that A_4 is not *before* A_2 .

Thus since A_4 cannot coincide with A_2 and since A_4 and A_2 both lie in the optical line a we must have A_4 *after* A_2 and so A_4 lies in α_2 .

Now let e be the optical line through A_3 parallel to c ; then e is an after-parallel of c , since A_3 is *after* A_1 .

Again, there is one single optical line (say f) through A_2 intersecting e in some element, say A_5 , which lies in α_2 .

Now since A_2 and A_3 are distinct elements both lying in α_1 , and since A_2 does not lie in the optical line A_1A_3 , it follows by Theorem 13 that A_2 is neither *before* nor *after* A_3 and therefore A_5 lies in α_3 .

Suppose now, if possible, that A_5 is distinct from A_4 ; then by Theorem 17 (a) since A_4 and A_5 lie both in α_2 and α_3 , the one is neither *before* nor *after* the other.

Thus A_5 could not lie either in a or d since then it would have to be either *before* or *after* A_4 .

Neither can A_5 lie in b , for since A_2 is *after* A_1 and A_5 is *after* A_2 , and A_1 is an element of b , it would then follow by Theorem 12 that A_2 lay in b , which is impossible.

Thus e is the only optical line through A_5 containing an element of b and if e also intersected a it would have to coincide with d , since d is the only optical line through A_3 which intersects a .

Thus if A_5 did not coincide with A_4 then A_5 could not lie in the acceleration plane defined by a and b .

Thus the acceleration plane defined by c and e would be distinct from the acceleration plane defined by a and b .

Now let g be the optical line through A_1 parallel to f ; then g is a before-parallel of f , since A_1 is *before* A_2 .

Then g could not coincide with b for in that case we should have two optical lines a and f both through A_2 and both parallel to b , which is impossible.

Now A_3 lies in the optical line b which intersects g in A_1 and so if A_3 lay in the acceleration plane defined by f and g then b would have to intersect f .

But the only optical line through A_1 intersecting f is c and so if A_3 lay in the acceleration plane defined by f and g then b would have to coincide with c , which is impossible.

Thus A_3 would not lie in this acceleration plane which therefore would be distinct from the acceleration planes defined by a and b and by c and e , which both contain A_3 .

But the acceleration planes defined by a and b , by c and e , and by f and g all contain the two elements A_1 and A_2 , while the two first-mentioned acceleration planes also contain A_3 , which would not be contained by the acceleration plane defined by f and g .

This is contrary to PoSt. XIII and so the assumption that A_5 is distinct from A_4 must be abandoned.

Thus A_5 coincides with A_4 and therefore the optical line d coincides with the after-parallel of c through A_3 .

This proves the theorem.

THEOREM 30.

If a, b, c, d , etc. be a set of parallel optical lines which all intersect one optical line l in elements A, B, C, D , etc., then through any element of one of the set of optical lines a, b, c, d , etc. other than the elements A, B, C, D , etc. there is one optical line which intersects each one of the set a, b, c, d , etc. and is parallel to l .

Since the elements A, B, C, D , etc. are elements of one optical line l , therefore of any two of these elements one is *after* the other by Theorem 9.

Thus of any two of the parallel optical lines a, b, c, d , etc. one is an after-parallel of the other.

If then one of these optical lines be selected (say b) and any element in it (say X) distinct from B there will be

one optical line through X intersecting a ,
one optical line through X intersecting c ,
one optical line through X intersecting d , etc.

But by Theorem 29 all these are parallel to l and since they all go through X they must all be identical.

Also for each element of b there is one such optical line and since any pair of such optical lines are parallel to l they are also parallel to one another.

This theorem shows that an acceleration plane contains two sets of parallel optical lines which may be called the *generators* of the acceleration plane.

Any generator of one set intersects every generator of the other set but does not intersect any one of its own set.

Also we see that through any element of an acceleration plane there are two optical lines lying in the acceleration plane and of these two one belongs to one set and the other to the other set.

THEOREM 31.

Three distinct elements cannot lie in pairs in three distinct optical lines.

Let A_1, A_2 and A_3 be three distinct elements and let A_1 and A_2 lie in one optical line.

We may suppose that it is A_2 which lies in α_1 .

If then A_1 and A_3 lie in another optical line we may suppose either that A_3 lies in α_1 or in β_1 .

First suppose A_3 lies in α_1 .

Then if A_3 and A_2 lie in an optical line one of them must be *after* the other and so by Theorem 13 (a) A_3 must lie in the optical line A_1A_2 .

Next suppose that A_3 lies in β_1 .

Then if A_3 and A_2 lie in one optical line, A_1 is *before* A_2 one element of it and *after* A_3 another element of it and so by Theorem 12 A_1 must lie in the optical line A_3A_2 . Thus the optical lines are not distinct and so the theorem is proved.

THEOREM 32.

Through any element of an acceleration plane there are only two distinct optical lines which lie in the acceleration plane.

We have already seen that there are two optical lines which pass through any element of an acceleration plane and lie in the acceleration plane.

We have now to prove that there cannot be more than two.

Let A_1 be any particular element of an acceleration plane and let a and b be the two generators of the acceleration plane passing through A_1 .

Suppose, if possible, that a third optical line c passes through A_1 and lies in the acceleration plane.

Let A_2 be an element of c *after* A_1 , then A_2 must lie in the acceleration plane and so there would be two generators of the acceleration plane passing through A_2 and parallel respectively to a and b .

The optical line parallel to a would meet b in some element, A_3 say, and the optical line parallel to b would meet a in A_4 say.

But if A_1 , A_2 and A_3 were all distinct we should have three elements lying in pairs in three distinct optical lines, which is impossible by Theorem 31.

Similarly if A_1 , A_2 and A_4 were all distinct.

Thus any optical line through A_1 and lying in the acceleration plane must coincide either with a or b .

THEOREM 33.

If an acceleration plane contain an optical line a and an element A_1 which does not lie in the optical line, then A_1 is either before or after an element of a .

There are two optical lines in the acceleration plane which pass through A_1 .

Of these two, one which we shall call b intersects a , while the other does not intersect it.

If b intersects a in an element A_2 , then A_2 must be distinct from A_1 since A_1 does not lie in a .

But both A_1 and A_2 lie in the optical line b and so the one is *after* the other.

Thus A_1 is either *before* or *after* A_2 : an element of the optical line a .

THEOREM 34.

If two elements be such that one is after the other, but does not lie in an optical line with it, then there are an infinite number of acceleration planes containing the two elements.

Let A_1 and A_2 be the two elements and let A_2 be *after* A_1 .

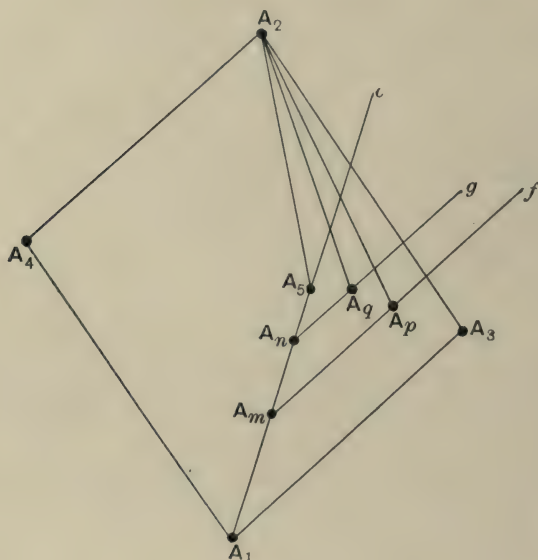


Fig. 7.

Then by Theorem 5 there is at least one other distinct element which is a member both of α_1 and of β_2 . Call such an element A_3 .

Then A_2 is in α_3 and so both A_1A_3 and A_3A_2 are optical lines.

But A_1 is not in the optical line A_3A_2 but is *before* A_3 an element of it and so we may take a before-parallel to A_3A_2 through A_1 .

Then through A_2 there is one single optical line intersecting this before-parallel in some element, say A_4 .

Then A_1A_2 will be an after-parallel of A_1A_3 by Theorem 29.

Now A_1A_3 and A_1A_4 are two distinct optical lines through A_1 and by Theorem 19 there are at least three distinct optical lines containing

A_1 so that there must be at least one other. Let c be such an optical line.

Then A_2 is not in c but is *after* A_1 an element of c and so by Post. IX (b) there is *one single element* (say A_5) common to the optical line c and the sub-set β_2 .

Then A_1A_5 and A_5A_2 are distinct optical lines and since A_2 is *after* A_5 we may take an after-parallel to A_1A_5 through A_2 , which together with A_1A_5 will determine an acceleration plane containing the given elements.

Let A_m and A_n be any two distinct elements of the optical line A_1A_5 which are *after* A_1 and *before* A_5 .

Then, A_m and A_n being elements which are *after* A_1 and not in the optical line A_1A_3 , we may take after-parallels to A_1A_3 through A_m and A_n . Call these f and g respectively.

Then A_2 cannot be an element of f for then we should have the three elements A_m , A_5 and A_2 lying in pairs in three distinct optical lines, which is impossible by Theorem 31.

But A_5 is *after* A_m and A_2 is *after* A_5 and so by Post. III A_2 is *after* A_m an element of f .

Thus by Post. IX (b) there is *one single element* (say A_p) common to the optical line f and the sub-set β_2 .

Similarly A_2 cannot be an element of g but is *after* A_n an element of g and so there is *one single element*, say A_q , common to the optical line g and the sub-set β_2 .

Now A_m and A_n being both elements of the optical line A_1A_5 , the one must be *after* the other, and since f and g are both after-parallels of A_1A_3 it follows by Theorem 23 that the one is an after-parallel of the other.

Thus f and g can have no element in common and so A_p and A_q must be distinct.

Further A_p and A_q cannot both lie in the same optical line through A_2 , for since f and g are both after-parallels of A_1A_3 therefore by Theorem 30 this hypothetical optical line would also intersect A_1A_3 and would therefore have to be identical with A_3A_2 . Thus the optical line A_1A_5 or c would have to be parallel to A_3A_2 and so be identical with A_1A_4 , contrary to hypothesis.

Thus the optical lines A_pA_2 and A_qA_2 must be distinct.

Further, either of them, say A_pA_2 , must be distinct from A_3A_2 for then A_3A_2 would contain A_p an element of f , and since f is an after-parallel of A_1A_3 therefore again A_1A_5 would have to be identical with A_1A_4 , contrary to hypothesis.

Again, either of the optical lines $A_p A_2$ or $A_q A_2$ must be distinct from $A_s A_2$, for, if we take $A_p A_2$, we should have the three elements A_m , A_p and A_s lying in pairs in three distinct optical lines, which is impossible.

Similarly corresponding to each element of the optical line $A_1 A_5$ which is *after* A_1 and *before* A_5 we may take an after-parallel to $A_1 A_5$ which will have one element in common with the sub-set β_2 which determines a distinct optical line through A_2 .

Since there are an infinite number of elements in the optical line $A_1 A_5$ which are *after* A_1 and *before* A_5 , it follows that there are an infinite number of optical lines through A_2 which are all distinct.

Since A_1 and A_2 are not in one optical line therefore A_1 cannot lie in any of these optical lines through A_2 .

But A_1 is *before* A_2 and so by Post. X (a) a before-parallel to each of these optical lines may be taken through A_1 and the pair of parallel optical lines will determine an acceleration plane containing A_1 and A_2 .

Also since the number of optical lines through A_2 is infinite, and since by Theorem 32 only two optical lines pass through any element of an acceleration plane and lie in the acceleration plane, it follows that there are an infinite number of acceleration planes containing the two elements A_1 and A_2 .

THEOREM 35.

If two distinct elements be such that the one is neither before nor after the other then there are an infinite number of acceleration planes containing the two elements.

Let A_1 and A_2 be the two elements.

Then by Post. VI (a) and Post. XI (a) there are at least two other distinct elements which are members both of α_1 and α_2 .

Let A_3 and A_5 be two such elements.

Then $A_1 A_3$, $A_1 A_5$, $A_2 A_3$, $A_2 A_5$ are distinct optical lines.

Let A_m and A_n be any two distinct elements of the optical line $A_1 A_5$ which are *after* A_1 and *before* A_5 .

Then A_m and A_n being elements which are *after* A_1 and not in the optical line $A_1 A_3$, we may take after-parallels to $A_1 A_3$ through A_m and A_n . Call these f and g respectively.

Then A_2 cannot be an element of f , for then we should have the three elements A_m , A_5 and A_2 lying in pairs in three distinct optical lines, which is impossible by Theorem 31.

But since f is an after-parallel of $A_1 A_3$ it follows by Theorem 20 (a) that $A_1 A_3$ is a before-parallel of f and so A_3 is *before* some element of f , or there is an element of f which is *after* A_3 .

But A_3 is *after* A_2 and so by Post. III there is an element of f which is *after* A_2 , or A_2 is *before* an element of f .

Thus by Post. IX (a) there is *one single element*, say A_p , which is an element both of the optical line f and the sub-set α_2 .

Similarly A_2 cannot be an element of g but is *before* an element of g and so there is *one single element*, say A_q , common to the optical line g and the sub-set α_2 .

Now A_m and A_n being both elements of the optical line A_1A_5 , the one must be *after* the other, and since f and g are both after-parallels of A_1A_3 it follows by Theorem 23 that the one is an after-parallel of the other.

Thus f and g can have no element in common and so A_p and A_q must be distinct.

Further, A_p and A_q cannot both lie in the same optical line through A_2 , for since f and g are both after-parallels of A_1A_3 it follows by Theorem 30 that this hypothetical optical line would also intersect A_1A_3 and would therefore have to be identical with A_2A_3 .

Thus A_2A_3 would, by Theorem 29, require to be either a before or after-parallel of A_1A_5 .

But A_3 is *after* A_1 and A_2 is *before* A_5 and so one element of A_2A_3 is *after* an element of A_1A_5 while another element of A_2A_3 is *before* an element of A_1A_5 .

Thus A_2A_3 cannot be either a before or after-parallel of A_1A_5 and so A_p and A_q cannot both lie in the same optical line through A_2 .

Thus the optical lines A_2A_p and A_2A_q must be distinct.

Further, either of them must be distinct from A_2A_3 , for otherwise A_2A_3 would, again, require to be an after-parallel of A_1A_5 , which we showed to be impossible.

Again, either of the optical lines A_2A_p , A_2A_q must be distinct from A_2A_5 , for if we take for instance the case of A_2A_p , we should then have the three elements A_m , A_p and A_5 lying in pairs in three distinct optical lines, which is impossible.

Similarly corresponding to each element of the optical line A_1A_5 which is *after* A_1 and *before* A_5 we may take an after-parallel to A_1A_3 which will have one element in common with the sub-set α_2 which determines a distinct optical line through A_2 .

Since there are an infinite number of elements in the optical line A_1A_5 which are *after* A_1 and *before* A_5 , it follows that there are an infinite number of optical lines through A_2 which are all distinct.

Since A_1 is neither *before* nor *after* A_2 and is distinct from it, therefore A_1 cannot lie in any of the optical lines through A_2 .

Now there is only one element common to the optical line A_1A_5 and the sub-set α_2 , namely the element A_5 , and by Theorem 1 (a) A_2 is *before* A_5 but not *before* any element outside the sub-set α_2 which is *before* A_5 .

Thus A_2 is not *before* A_m or A_n .

But A_2 is *before* A_p and *before* A_q , and so we see that A_p must be *after* A_m and A_q *after* A_n .

But A_m and A_n are both *after* A_1 and so by Post. III A_p is *after* A_1 and A_q is *after* A_1 .

Thus A_1 is not an element of any of the optical lines through A_2 but is *before* elements of those which we have obtained, and so by Post. X (a) there is *one single optical line* containing A_1 and such that each element of it is *before* an element of any particular one of the optical lines through A_2 which we have obtained.

Each of these pairs of parallel optical lines determines an acceleration plane containing A_1 and A_2 and, since the number of optical lines through A_2 is infinite, and since by Theorem 32 there are only two optical lines which pass through any element of an acceleration plane and lie in the acceleration plane, it follows that there are an infinite number of acceleration planes containing the two elements A_1 and A_2 .

REMARKS.

The last two theorems showed that an infinite number of acceleration planes contain any pair of elements which *do not* lie in an optical line.

Further, Theorem 28 showed that an infinite number of acceleration planes contain a given optical line and so contain any two elements which *do* lie in an optical line.

It is easy to show that if two or more distinct acceleration planes contain an optical line there is no other element which they have in common which does not lie in the optical line.

Thus if we consider two acceleration planes P and Q which both contain an optical line a , and suppose, if possible, that they have also an element A in common which does not lie in the optical line, then another optical line b through A must exist which is parallel to a .

The optical line b must lie in the acceleration plane P and also in the acceleration plane Q , and b must be either a *before* or *after*-parallel of a , since A is either *before* or *after* an element of a (Theorem 33).

Thus a and b determine an acceleration plane which would be identical both with P and Q , which could therefore not be distinct, contrary to hypothesis.

Thus if two or more acceleration planes have an optical line in

common they can have no other element outside the optical line in common.

We have also seen by Post. XIII that any acceleration plane which contains two elements which are common to two distinct acceleration planes, contains all elements common to them.

These remarks prepare the way for the following definitions and for Postulate XIV.

Definitions. If two acceleration planes contain two elements in common, then the aggregate of all elements common to the two acceleration planes will be called a *general line*.

If two acceleration planes contain two elements in common, of which one is *after* the other, but does not lie in the same optical line with it, then the aggregate of all elements common to the two acceleration planes will be called an *inertia line**.

If two acceleration planes contain two elements in common, of which one is neither *before* nor *after* the other, then the aggregate of all elements common to the two acceleration planes will be called a *separation line**.

POSTULATE XIV. (a) **If a be any inertia line and A_1 be any element of the set, then there is one single element common to the inertia line a and the sub-set α_1 .**

(b) **If a be any inertia line and A_1 be any element of the set, then there is one single element common to the inertia line a and the sub-set β_1 .**

THEOREM 36.

An inertia line in any acceleration plane has one single element in common with each optical line in the acceleration plane.

Let a be the inertia line and let A_1 be an element in any optical line b in the acceleration plane.

Then by Post. XIV (a) there is *one single element*, say A_2 , common to the inertia line a and the sub-set α_1 .

Also by Post. XIV (b) there is *one single element*, say A_3 , common to the inertia line a and the sub-set β_1 .

Now if A_1 lay in a , both A_2 and A_3 must coincide with A_1 since, if

* The name "inertia line" has been adopted because an inertia line represents the time path of an unaccelerated particle.

The name "separation line" has been adopted because a single particle cannot occupy more than one element of a separation line, so that if particles P and Q occupy distinct elements of a separation line they must be *separate* particles.

A_2 were distinct from A_1 we should have the two elements A_1 and A_2 in a which both lay in α_1 , contrary to Post. XIV (a) which asserts that there is only *one single element* common to the inertia line a and the sub-set α_1 .

Thus if A_1 lie in a , then A_2 must coincide with A_1 .

Similarly if A_1 lie in a , then A_3 must coincide with A_1 .

Suppose now that A_1 does not lie in a , then both A_2 and A_3 must be distinct from A_1 .

Further A_3 cannot lie in β_2 for then by Post. VII (b) A_2 would be a member of α_3 and so we should have the two elements A_2 and A_3 both common to the inertia line a and the sub-set α_3 , contrary to Post. XIV (a).

Now since A_1 , A_2 and A_3 are distinct we must have A_2 *after* A_1 and A_1 *after* A_3 and therefore A_2 *after* A_3 .

Further, since A_2 does not lie in α_3 it follows that A_2 and A_3 are not in the same optical line.

But since A_2 is in α_1 and A_3 in β_1 it follows that A_1 and A_2 lie in an optical line through A_1 , and also A_3 and A_1 lie in an optical line through A_1 , and these optical lines are distinct.

Now by Theorem 32 there are only two distinct optical lines in the acceleration plane which pass through A_1 , and so one of them must be A_1A_2 and the other A_3A_1 , and since b must be identical with one of these optical lines, it follows that a and b must have one single element in common.

THEOREM 37.

Of any two distinct elements of an inertia line one is after the other.

Let A_1 and A_2 be any two distinct elements of the inertia line a , and let b be one of the two optical lines in an acceleration plane containing a which pass through A_1 .

Now of the two optical lines in this acceleration plane which pass through A_2 , the one is parallel to b and the other intersects it in some element, say A_3 .

Now A_1 and A_2 being distinct cannot both lie in α_3 by Post. XIV (a) and they cannot both lie in β_3 by Post. XIV (b).

Thus one of the two elements A_1 and A_2 must lie in α_3 and the other in β_3 , and so one of them must be *after* A_3 and the other *before* A_3 .

Thus by Post. III one of the two elements A_1 and A_2 must be *after* the other.

From the definition of a separation line it contains a pair of elements one of which is neither *before* nor *after* the other.

Thus it follows from the above theorem that *no inertia line can be a separation line and no separation line can be an inertia line.*

THEOREM 38.

If A_1 be any element in an inertia line a , there is at least one other element in the inertia line which is after A_1 and also at least one other element in it which is before A_1 .

Let b be one of the two optical lines through A_1 in any acceleration plane which contains a and let A_2 be any element in b which is *after* A_1 .

Then by Post. XIV (a) there is one single element, say A_3 , common to the inertia line a and the sub-set α_2 .

Then A_3 cannot be identical with A_2 since then we should have two elements common to the inertia line a and the optical line b , contrary to Theorem 36.

Thus A_3 is *after* A_2 and A_2 is *after* A_1 and therefore A_3 is *after* A_1 and is an element of the inertia line a .

Similarly if we take any element A_4 in the optical line b and *before* A_1 there will by Post. XIV (b) be one single element, say A_5 , common to the inertia line a and the sub-set β_4 .

Then A_1 will be *after* A_4 and A_4 *after* A_5 and therefore A_1 *after* A_5 .

Thus A_5 is *before* A_1 and is an element of the inertia line a .

THEOREM 39.

If A_1 and A_2 be any two distinct elements of an inertia line a , there is at least one other distinct element of a which is after one of the two elements and before the other.

By Theorem 37 one of the two elements A_1 and A_2 is *after* the other.

We shall suppose that A_2 is *after* A_1 .

Let b and c be the two optical lines through A_1 in any acceleration plane containing a .

Then the after-parallel to c through A_2 will intersect b in some element A_3 which must be *after* A_1 while the after-parallel to b through A_2 will intersect c in some element A_4 *after* A_1 .

Now take any element A_5 in b *after* A_1 and *before* A_3 and take the optical line through A_5 parallel to c .

This parallel through A_5 will intersect the optical line A_4A_2 in some element A_6 .

Then A_5A_6 is an after-parallel of A_1A_4 but a before-parallel of A_3A_2 , and so A_4 is *before* A_2 .

But by Theorem 36 there is one single element (say A_7) common to the inertia line a and the optical line A_5A_6 and this element must be distinct from A_1 and A_2 .

Also by Post. XIV there is one single element common to the inertia line a and the sub-set α_5 and one single element common to the inertia line a and the sub-set β_5 and since A_1 is *before* A_5 and so lies in β_5 , it follows that A_7 must lie in α_5 .

Similarly A_7 must lie in β_6 .

As A_7 cannot be identical with A_5 or A_6 and since it lies in α_5 and in β_6 , it follows that A_7 is *after* A_5 and A_6 is *after* A_7 .

Thus since A_5 is *after* A_1 , therefore by Post. III A_7 is *after* A_1 , and since A_2 is *after* A_6 , therefore A_2 is *after* A_7 .

Thus A_7 is *after* A_1 and *before* A_2 and lies in the inertia line a .

It follows from the above results that there are an infinite number of elements in any inertia line.

POSTULATE XV. If two general lines, one of which is a separation line and the other is not, lie in the same acceleration plane, then they have an element in common.

Since there are an infinite number of optical lines in an acceleration plane, and since only two of them pass through any given element, and since by Post. XV each of them has an element in common with any separation line lying in the acceleration plane, it follows that there are an infinite number of elements in any separation line.

Further, since as we have remarked in connection with Theorem 37 no inertia line can be a separation line, it follows that *no element of a separation line is either before or after another element of it.*

THEOREM 40.

If A_1 and A_2 be two distinct elements one of which is neither before nor after the other, and if a and b be the two optical lines through A_1 in an acceleration plane containing the two elements, then A_2 is before an element of one of these optical lines and after an element of the other.

By Theorem 33 A_2 must be either *before* or *after* an element of a and also must be either *before* or *after* an element of b ; but A_2 cannot lie either in a or b since it is distinct from A_1 and is neither *before* nor *after* it.

Suppose first that A_2 is *before* an element of a .

Then one of the two optical lines through A_2 in the acceleration plane will intersect a in some element, say A_3 , while the other optical line through A_2 in the acceleration plane will intersect b in some element, say A_4 .

Then A_2 must be *before* A_3 since A_2 cannot either lie in a or be *after* any element of it.

But A_3 cannot either coincide with A_1 or be *before* A_1 , for then we should have A_2 *before* A_1 , contrary to hypothesis.

Thus A_3 must be *after* A_1 .

But A_1 is an element of b and so the optical line A_2A_3 (which since it intersects a must be parallel to b) must be an after-parallel of b .

Thus A_2 must be *after* an element of b , and since A_2 must be either *before* or *after* A_4 , it follows that A_2 is *after* A_4 .

In a similar manner we may prove that if A_2 be *before* an element of b it must be *after* an element of a .

Also in an analogous manner we may show that if A_2 be *after* an element of b it must be *before* an element of a , and if A_2 be *after* an element of a it must be *before* an element of b .

Thus A_2 must be *before* an element of one of the optical lines a and b and *after* an element of the other.

Definition. An element in an acceleration plane will be said to be *between* a pair of parallel optical lines in the acceleration plane if it be *after* an element of the one optical line and *before* an element of the other and does not lie in either optical line.

THEOREM 41.

If A_1 and A_2 be any two distinct elements of a separation line, there is at least one other element of the separation line which lies between a pair of parallel optical lines through A_1 and A_2 respectively in an acceleration plane containing the separation line.

Let a_1 and b_1 be the two optical lines passing through A_1 in any acceleration plane containing the separation line.

Then, since A_2 is neither *before* nor *after* A_1 , it follows that A_2 is *before* an element of one of the two optical lines a_1 and b_1 and is *after* an element of the other. (Theorem 40.)

Suppose that A_2 is *before* an element of a_1 .

Then it is *after* an element of b_1 .

Let a_2 and b_2 be the two optical lines through A_2 parallel respectively to a_1 and b_1 .

Then a_2 and b_2 lie in the acceleration plane and since A_2 is *before* an element of a_1 therefore a_2 is a before-parallel of a_1 .

Similarly since A_2 is *after* an element of b_1 it follows that b_2 is an after-parallel of b_1 .

Further b_2 must intersect a_1 in some element, say A_3 , which must be *after* A_2 since a_1 is an after-parallel of a_2 .

Let A_4 be any element of b_2 which is *after* A_2 and *before* A_3 and consider the optical line through A_4 parallel to a_1 .

We shall call this optical line a' .

Then since A_4 is *before* A_3 it follows that a' is a before-parallel of a_1 and since A_4 is *after* A_2 therefore a' is an after-parallel of a_2 .

Also a' lies in the acceleration plane.

Thus by Post. XV a' must have an element in common with the separation line A_1A_2 .

Call this element A_5 .

Then since a' is a before-parallel of a_1 therefore A_5 is *before* an element of a_1 and since a' is an after-parallel of a_2 therefore A_5 is *after* an element of a_2 .

Thus A_5 is between the parallel optical lines a_1 and a_2 .

THEOREM 42.

If A_1 , A_2 and A_3 be three elements in a separation line and if A_3 lies between a pair of parallel optical lines through A_1 and A_2 in an acceleration plane containing the separation line, then A_3 also lies between a second pair of parallel optical lines through A_1 and A_2 in the acceleration plane.

Let a_1 and a_2 be a pair of parallel optical lines through A_1 and A_2 respectively in the acceleration plane and suppose that A_3 lies between them.

We may without loss of generality suppose that A_3 is *after* an element of a_2 and *before* an element of a_1 .

Let b_1 be the second optical line which passes through A_1 in the acceleration plane and let b_2 be the second optical line which passes through A_2 in the acceleration plane.

Then, since a_1 and a_2 are parallel, b_1 and b_2 are also parallel.

But since A_3 and A_1 lie in a separation line, A_3 is neither *before* nor *after* A_1 and since A_3 is *before* an element of a_1 therefore, by Theorem 40, A_3 is *after* an element of b_1 .

Similarly A_3 is neither *before* nor *after* A_2 and, since A_3 is *after* an element of a_2 , therefore, by Theorem 40, A_3 is *before* an element of b_2 .

Thus A_3 is between the parallel optical lines b_1 and b_2 passing through A_1 and A_2 respectively in the acceleration plane.

Since there are only two optical lines in an acceleration plane which pass through a given element of it, it follows directly from the above theorem that *if A_1 , A_2 and A_3 be three elements in a separation line and if A_3 lies between a pair of parallel optical lines through A_1 and A_2 in an acceleration plane containing the separation line, then A_2 does not lie between a pair of parallel optical lines through A_1 and A_3 in the acceleration plane.*

Similarly A_1 does not lie between a pair of parallel optical lines through A_2 and A_3 in the acceleration plane.

THEOREM 43.

If A_1 and A_2 be any two elements of a separation line, there is at least one other element of the separation line such that A_2 lies between a pair of parallel optical lines through A_1 and that element is an acceleration plane containing the separation line.

Using the notation employed in Theorem 41 let us take any element, say A_6 , in the optical line b_2 and *before* A_2 and consider the optical line through A_6 parallel to a_2 .

Call this optical line a'' .

Then since A_6 is *before* A_2 therefore a'' is a before-parallel of a_2 , and since a_2 is a before-parallel of a_1 therefore a'' is also a before-parallel of a_1 .

Further a'' lies in the acceleration plane and so by Post. XV it has an element in common with the separation line.

Call this element A_7 .

Then A_2 is *before* A_3 an element of a_1 and is *after* A_6 an element of a'' .

Thus A_2 is between the parallel optical lines a_1 and a'' passing through A_1 and the element A_7 respectively and lying in the acceleration plane.

THEOREM 44.

Of any three distinct elements of a separation line in a given acceleration plane there is one which lies between a pair of parallel optical lines through the other two and in the acceleration plane.

Let A_1 , A_2 and A_3 be any three distinct elements in the separation line.

Then since there are two optical lines in an acceleration plane passing through any element of it, let us select one of those passing through one of these elements, say A_1 , and the parallel optical lines through A_2 and A_3 .

Call these optical lines a_1 , a_2 and a_3 respectively.

Then a_1 , a_2 and a_3 all intersect any generator of the acceleration plane belonging to the opposite set.

Let b be such a generator and suppose that a_1 , a_2 and a_3 intersect b in the elements A_1' , A_2' and A_3' respectively.

Then A_1' , A_2' and A_3' being all elements of the optical line b , and being all distinct, it follows that of any two of them one must be *after* the other.

Thus remembering that Post. III must be satisfied it follows that either

$$\begin{array}{ll}
 & A_2' \text{ is after } A_1' \text{ and } A_3' \text{ after } A_2' \quad (1) \} \\
 \text{or} & A_2' \text{ is after } A_3' \text{ and } A_1' \text{ after } A_2' \quad (2) \} \\
 \text{or} & A_3' \text{ is after } A_1' \text{ and } A_2' \text{ after } A_3' \quad (3) \} \\
 \text{or} & A_3' \text{ is after } A_2' \text{ and } A_1' \text{ after } A_3' \quad (4) \} \\
 \text{or} & A_1' \text{ is after } A_2' \text{ and } A_3' \text{ after } A_1' \quad (5) \} \\
 \text{or} & A_1' \text{ is after } A_3' \text{ and } A_2' \text{ after } A_1' \quad (6) \}
 \end{array}$$

In case (1) a_2 is an after-parallel of a_1 and a before-parallel of a_3 and so each element of a_2 is between the parallel optical lines a_1 and a_3 .

Thus A_2 is between a pair of parallel optical lines through A_1 and A_3 in the acceleration plane.

Similarly in case (2) a_2 is an after-parallel of a_3 and a before-parallel of a_1 and therefore again A_2 is between a pair of parallel optical lines through A_1 and A_3 in the acceleration plane.

In a similar manner in cases (3) and (4) A_3 is between a pair of parallel optical lines through A_1 and A_2 in the acceleration plane; while in cases (5) and (6) A_1 is between a pair of parallel optical lines through A_2 and A_3 in the acceleration plane.

Thus in all cases one of the three elements is between a pair of parallel optical lines through the other two and in the acceleration plane.

THEOREM 45.

If A be an element of an optical line a and if B be an element which is neither before nor after any element of a , then no element of the general line AB , with the exception of A , is either before or after any element of a .

Let C be any element of the general line AB other than A , and let c be an optical line through C parallel to a .

Suppose, if possible, that C is either *before* or *after* some element of a .

Then c would be either a before or after-parallel of a and accordingly c and a would be generators of an acceleration plane which would contain the two elements A and C of the general line AB and would therefore contain every element of AB .

Thus the element B would lie in an acceleration plane containing the optical line a , and therefore, by Theorem 33, B would be either *before* or *after* an element of a , contrary to hypothesis.

Thus the assumption that any element of the general line AB , other than A , is either *before* or *after* any element of a leads to a contradiction and therefore is not true and so no element of AB with the exception of A is either *before* or *after* any element of a .

SETS OF THREE ELEMENTS WHICH DETERMINE ACCELERATION PLANES.

Let A_1 , A_2 and A_3 be three distinct elements which do not all lie in one general line, then A_1 and A_2 must lie in one general line, A_2 and A_3 in a second and A_3 and A_1 in a third.

These three general lines need not however lie in one acceleration plane, although they do in certain cases.

In these latter cases the three elements determine the acceleration plane containing them, since if they lay in two distinct acceleration planes they would lie in one general line, contrary to hypothesis.

It is important to have criteria by which we can say that a set of three elements does lie in one acceleration plane.

CASE I. Three elements A_1 , A_2 , A_3 lie in one acceleration plane if A_1 and A_2 lie in an optical line while A_3 is an element not in the optical line but *before* some element of it, or *after* some element of it.

This is clearly true, since, if A_1 and A_2 lie in the optical line a , while A_3 does not lie in a but is *before* some element of it, then there is a before-parallel optical line, say b , containing A_3 and so a and b are a pair of parallel generators of an acceleration plane, containing A_1 , A_2 and A_3 and which is determined by them.

Similarly if A_3 be *after* some element of a there is a definite after-parallel optical line b containing A_3 and so the two optical lines a and b are a pair of parallel generators of an acceleration plane containing A_1 , A_2 and A_3 and which is determined by them.

CASE II. Three elements A_1 , A_2 , A_3 lie in one acceleration plane if A_1 and A_2 lie in an inertia line and A_3 be *any* element outside the inertia line.

This can also be readily seen to hold since if a denote the inertia

line containing A_1 and A_2 then by Post. XIV (a) there is one single element, say A_4 , common to the inertia line a and the sub-set α_3 , and by Post. XIV (b) there is one single element, say A_5 , common to the inertia line a and the sub-set β_3 .

Thus A_3 and A_4 lie in one optical line while A_3 and A_5 lie in another optical line.

These two optical lines may be taken as generators of opposite sets of an acceleration plane containing A_3 , A_4 and A_5 .

But since this acceleration plane contains the two elements A_4 and A_5 of the inertia line a , it must contain every element of a and therefore contains A_1 and A_2 .

Thus the three elements A_1 , A_2 and A_3 lie in one acceleration plane which is determined by them.

CASE III. Three elements A_1 , A_2 , A_3 lie in one acceleration plane if A_1 and A_2 lie in a separation line and if A_3 be an element not in the separation line but *before* at least two elements of it or *after* at least two elements of it.

In order to show this let a be the separation line containing A_1 and A_2 and suppose A_3 is *before* the elements A_4 and A_5 of a which are supposed distinct.

Then A_3 and A_4 must lie either in an optical line or an inertia line since A_4 is *after* A_3 , and similarly A_3 and A_5 must lie either in an optical line or an inertia line and the two general lines A_3A_4 and A_3A_5 are distinct.

If A_3A_4 and A_3A_5 be both optical lines, then they may be taken as generators of opposite sets of an acceleration plane containing A_3 , A_4 and A_5 .

But this acceleration plane, since it contains the two distinct elements A_4 and A_5 of the separation line a , must contain every element of it and so must contain A_1 and A_2 .

Thus A_1 , A_2 and A_3 lie in one acceleration plane which is determined by them.

We shall suppose next that at least one of the general lines A_3A_4 and A_3A_5 is an inertia line.

Let us say that A_3A_4 is an inertia line.

Then by Case II the three elements A_3 , A_4 and A_5 lie in one acceleration plane which is determined by them.

But since this acceleration plane contains the two elements A_4 and A_5 of the separation line a , therefore it contains every element of a and so must contain A_1 and A_2 .

Thus A_1 , A_2 and A_3 lie in one acceleration plane which is determined by them.

The case when A_3 is *after* two distinct elements of a is quite analogous.

If A_1 and A_2 lie in an optical line a while A_3 is an element which is neither *before* nor *after* any element of a , then the three elements do not lie in one acceleration plane, for by Theorem 45 no element of the general line A_1A_3 with the exception of A_1 is either *before* or *after* any element of a .

But if A_1 , A_2 and A_3 lay in an acceleration plane there would be two optical lines through A_2 in the acceleration plane and both of these would have an element in common with the separation line A_1A_3 .

Thus there would be at least two elements of A_1A_3 which would be *before* or *after* A_2 , contrary to Theorem 45.

Thus A_1 , A_2 and A_3 do not lie in one acceleration plane.

If A_1 and A_2 lie in a separation line a , while A_3 is *before* one *single* element of a or *after* one *single* element of a , then the three elements A_1 , A_2 , A_3 cannot lie in one acceleration plane.

This is easily seen, for if we suppose that they do all lie in one acceleration plane, there are two optical lines through A_3 in the acceleration plane which have each an element in common with a .

If these elements be called A_4 and A_5 then, since a is a separation line, A_4 is neither *before* nor *after* A_5 and so A_4 and A_5 must be either both *before* or both *after* A_3 , contrary to the hypothesis that there is only one single element of a which A_3 is *after* or *before*.

If A_1 and A_2 lie in a separation line a , while A_3 does not lie in a and is neither *before* nor *after* any element of a , it is also evident from the above considerations that the three elements A_1 , A_2 , A_3 cannot lie in one acceleration plane.

We have not however as yet proved the possibility of this last case, but shall do so hereafter (Theorem 99). Till then no use will be made of it, and it is merely mentioned here for the sake of completeness.

Definition. If an acceleration plane have its two sets of generators respectively parallel to the two sets of generators of another distinct acceleration plane, then the two acceleration planes will be said to be *parallel* to one another.

It is clear that if P be an acceleration plane and A be any element outside it, then there is one single acceleration plane containing A and parallel to P ; for there is one single optical line through A parallel to

the one set of generators of P and one single optical line through A parallel to the other set of generators of P .

These are generators of opposite sets of an acceleration plane containing A and determine that acceleration plane, which is therefore unique.

It is further clear that two parallel acceleration planes can have no element in common, for if the element A lies outside the acceleration plane P and if a be an optical line passing through A and parallel to a generator of P , then a can have no element in common with P since otherwise it would require to lie entirely in P , contrary to the hypothesis that A is outside P .

Similarly any optical line b which intersects a and is parallel to a generator of P of the opposite set can have no element in common with P .

But if Q be the acceleration plane passing through A and parallel to P , every element of Q must lie in an optical line such as b and so P and Q can have no element in common.

It is also clear from the definition that *two distinct acceleration planes which are parallel to the same acceleration plane are parallel to one another*; since distinct optical lines which are parallel to the same optical line are parallel to one another.

THEOREM 46.

If an acceleration plane P have one element in common with each of a pair of parallel acceleration planes Q and R then, if P have a second element in common with Q , it has also a second element in common with R .

If P and Q have two elements in common they must have a general line in common which we may call a .

Let B_1 be the element which by hypothesis P and R have in common.

Then if a be an inertia or separation line it follows by Theorem 36 and Post. XV that both the optical lines through B_1 in the acceleration plane P have an element in common with a , while if a be an optical line one of the optical lines through B_1 in P has an element in common with a .

Thus in all cases at least one of the optical lines through B_1 in the acceleration plane P has an element in common with a .

Let A_1 be such an element.

Suppose first that a is an optical line.

Then a is one of the generators of Q and since the acceleration

plane R is parallel to Q and since B_1 lies in R there will be one of the generators of R passing through B_1 and parallel to a .

Since A_1 and B_1 lie in an optical line and are distinct, the one must be *after* the other and so this parallel to a through B_1 must be either a before or after-parallel of a .

Let us denote it by b .

Then a and b determine an acceleration plane which contains three distinct elements of P which are not all in one general line and so this acceleration plane must be identical with P .

Thus since it contains the optical line b it follows that P has a second element in common with R .

Suppose next that a is an inertia or separation line and let c be one of the generators of Q which pass through A_1 .

Then since R is parallel to Q and since B_1 lies in R there will be one of the generators of R passing through B_1 and parallel to c .

Since A_1 and B_1 lie in an optical line and are distinct, the one must be *after* the other and so this parallel to c through B_1 must be a before or after-parallel.

Let C be any element of c distinct from A_1 and let an optical line through C intersect the optical line through B_1 parallel to c in the element D .

Then by Theorem 29 the optical line CD must be a before or after-parallel of the optical line A_1B_1 .

Let the second optical line through C in the acceleration plane Q meet a in the element A_2 .

The element A_2 must exist since a is an inertia or separation line.

Since the optical line CA_2 must be a generator of Q of the opposite set to c , there must be an optical line through D in the acceleration plane R which is parallel to CA_2 and is a generator of R of the opposite set to DB_1 .

Since C and D lie in an optical line and are distinct, the one must be *after* the other and so this parallel to CA_2 through D must be a before or after-parallel.

Let an optical line through A_2 intersect the optical line through D parallel to CA_2 in the element B_2 .

Then by Theorem 29 the optical line A_2B_2 must be a before or after-parallel of CD and CD is a before or after-parallel of A_1B_1 and so if A_1B_1 and A_2B_2 be distinct they must be parallel to one another.

Now the optical lines CA_1 and CA_2 are distinct from the inertia or separation line a and are also distinct from one another.

Also the element C cannot lie in a since then CA_1 would have to be an inertia or separation line.

Thus the elements A_1 and A_2 are distinct and since they lie in an inertia or separation line they cannot lie in one optical line.

Thus A_2B_2 is distinct from A_1B_1 and is therefore parallel to it.

Also since the general line a and the optical line A_1B_1 lie in the acceleration plane P and since the element A_2 does not lie in A_1B_1 it follows by Theorem 33 that A_2 is either *before* or *after* some element of A_1B_1 .

Thus A_2B_2 must be either a before or after-parallel of A_1B_1 and so the optical lines A_1B_1 and A_2B_2 lie in an acceleration plane containing the general line a and the element B_1 .

This acceleration plane must therefore be identical with P and it contains the element B_2 in common with R where B_2 is distinct from B_1 .

Thus the theorem holds in all cases.

REMARKS.

It follows directly from this theorem that if two distinct acceleration planes P and Q have a general line in common and, if further, P has one element in common with an acceleration plane R which is parallel to Q , then P and R have a general line in common.

Further, since Q and R can have no element in common, it follows that these two general lines have no element in common.

Again if Q and R be two parallel acceleration planes and if a be any general line in Q , then there is at least one acceleration plane containing a and another general line in R .

This may be shown in the following way :

Let A_1 be any element of a and let f be any inertia line in R .

Then by Post. XIV (a) there is one single element common to the inertia line f and the sub-set α_1 . Let B be this element and let A_2 be any element of f which is *after* B .

Then A_2 is *after* A_1 but does not lie in α_1 and so A_1 and A_2 lie in an inertia line.

Thus A_2 and a lie in an acceleration plane, say P , which by Theorem 46 must contain a second element in common with R .

Thus P contains a and another general line in R .

It is easy to see that there are really an infinite number of acceleration planes which have this property of P .

We have seen that if two distinct optical lines intersect a pair of optical lines one of which is an after-parallel of the other, then of the

two first-mentioned optical lines one is an after-parallel of the other (Theorem 29).

We have also seen that it is impossible for an optical line to intersect a pair of neutral-parallel optical lines.

Thus we may state the following definition.

Definition. If two distinct optical lines intersect a pair of optical lines one of which is an after-parallel of the other, then the four optical lines will be said to form an *optical parallelogram*.

It is evident that an optical parallelogram must lie in an acceleration plane.

The elements of intersection will be spoken of as the *corners* of the optical parallelogram.

A pair of corners which lie in one optical line will be spoken of as *adjacent*.

A pair of corners which do not lie in one optical line will be spoken of as *opposite*.

A general line passing through a pair of opposite corners of an optical parallelogram will be spoken of as a *diagonal line* of the optical parallelogram.

We make a distinction between two optical parallelograms having a *diagonal line* in common and having a *diagonal* in common.

When we speak of two optical parallelograms having a *diagonal line* in common we shall mean that a pair of opposite corners of each of the optical parallelograms lie in the same general line.

When, on the other hand, we speak of two optical parallelograms having a *diagonal* in common, we mean that they have a pair of opposite corners in common.

It is obvious that an optical parallelogram has two diagonal lines and it is easy to see that *one of these must be an inertia line, and the other a separation line*.

For if we call the four optical lines a, b, c and d , and if a be an after-parallel of b while c is an after-parallel of d , then the intersection element of a and c must be *after* the intersection element of d and b so that these two intersection elements lie in an inertia line.

Further if we denote the intersection element of a and c by A_1 , that of a and d by A_2 , that of c and b by A_3 and that of d and b by A_4 , it follows by Theorem 13 (b) that if A_3 were either *before* or *after* A_2 then A_3 would have to lie in the optical line A_2A_1 or a , contrary to hypothesis.

Thus A_3 is neither *before* nor *after* A_2 and so A_2 and A_3 lie in a separation line.

Definition. If a general line a have one single element in common with a general line b , then a will be said to *intersect* b .

Since a general line does not intersect itself and since we may have two optical parallelograms in the same acceleration plane having a diagonal line in common, it is permissible to speak of two optical parallelograms in the same acceleration plane whose diagonal lines of one kind or the other do not intersect.

This prepares the way for Postulate XVI.

POSTULATE XVI. If two optical parallelograms lie in the same acceleration plane, then if their diagonal lines of one kind do not intersect, their diagonal lines of the other kind do not intersect.

THEOREM 47.

If a be any general line in an acceleration plane P and A_1 be any element of the acceleration plane which is not in the general line, then there is one single general line through A_1 in the acceleration plane which does not intersect a .

Let Q be any other acceleration plane distinct from P and containing the general line a , and let R be an acceleration plane passing through A_1 and parallel to Q .

Then by Theorem 46 P and R will have a general line in common which can have no element in common with a , and so there is at least one general line through A_1 in the acceleration plane P which does not intersect a .

We must next show that there is only one such general line.

Consider first the case where a is an optical line.

Then of the two optical lines through A_1 in the acceleration plane P we know that one is parallel to a while the other intersects it.

Further by Theorem 36 any inertia line through A_1 in the acceleration plane P must intersect a .

Also by Post. XV any separation line through A_1 in the acceleration plane P must intersect a .

Thus if a be an optical line there is one single general line through A_1 in the acceleration plane P which does not intersect a .

Consider next the cases where a is an inertia or a separation line.

If a be an inertia line, then by Theorem 36 both the optical lines through A_1 in the acceleration plane P intersect a , while by Post. XV every separation line in P intersects A_1 .

Thus when a is an inertia line any general line through A_1 in the acceleration plane P which does not intersect a can only be an inertia line.

Also from Post. XV it follows that when a is a separation line any general line through A_1 in the acceleration plane P which does not intersect a can only be a separation line.

With these provisos the demonstration of the unique character of the non-intersecting general line is similar in the two cases.

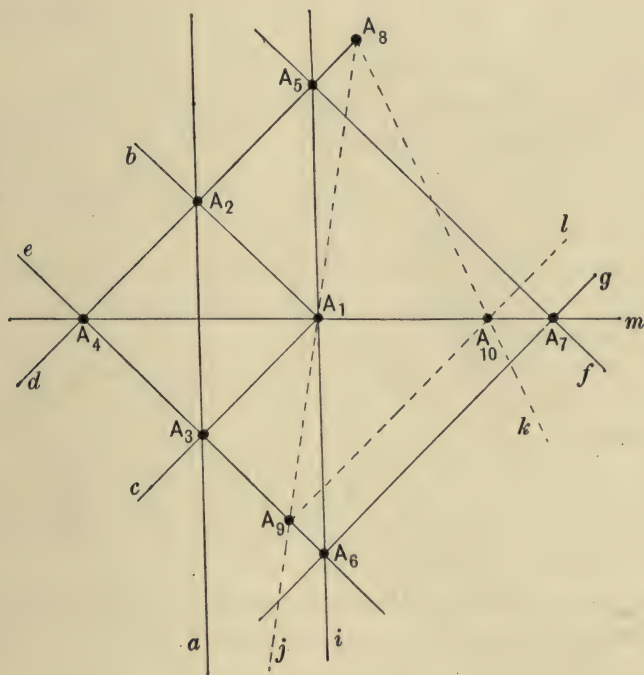


Fig. 8.

Suppose, if possible, that there are two general lines through A_1 in the acceleration plane, say i and j , which do not intersect a .

Then i and j must intersect in A_1 .

Let b and c be the two optical lines through A_1 in the acceleration plane and let them intersect a in A_2 and A_3 respectively.

Let d be the second optical line through A_2 in the acceleration plane and let e be the second optical line through A_3 in the acceleration plane and let d and e intersect in A_4 .

Then the optical lines b , c , d and e form an optical parallelogram.

Let m be the diagonal line through A_1 and A_4 .

Let the optical line d intersect i in A_5 and let the optical line e intersect i in A_6 .

Let f be the second optical line through A_5 in the acceleration plane and let g be the second optical line through A_6 in the acceleration plane and let f and g intersect in A_7 .

Then the optical lines f , g , d and e form an optical parallelogram and the diagonal line i is of the same kind as the diagonal line a of the optical parallelogram formed by b , c , d and e .

Thus since the diagonal lines a and i do not intersect it follows by Post. XVI that the diagonal lines of the other kind of the two optical parallelograms do not intersect.

But the two optical parallelograms have the corner A_4 in common and so they must have the diagonal line through A_4 in common.

Thus A_7 must lie in m .

Now suppose that the optical line d intersects j in A_8 and that the optical line e intersects j in A_9 .

Let k be the second optical line through A_8 in the acceleration plane and let l be the second optical line through A_9 in the acceleration plane and let k and l intersect in A_{10} .

Then the optical lines k , l , d and e form an optical parallelogram and since j is supposed not to intersect a it follows as before that A_{10} must lie in m .

But now we have the optical parallelograms formed by f , g , d and e , and by k , l , d and e having the diagonal line m in common, and so, by Post. XVI, their other diagonal lines do not intersect, which is contrary to the hypothesis that i and j intersected in A_1 .

Thus the hypothesis that there are two general lines through A_1 in the acceleration plane which do not intersect a leads to a contradiction and therefore is not true.

Thus there is in all cases *one single general line* through A_1 in the acceleration plane which does not intersect a .

THEOREM 48.

If two acceleration planes P and Q have a general line a in common, and if A_1 be any element which does not lie either in P or Q , then the acceleration planes through A_1 parallel to P and Q respectively have a general line in common.

Let R and S be the acceleration planes through A_1 parallel to P and Q respectively.

Two possibilities are open: either

- (1) Q has one element at least in common with R ,
or (2) Q has no element in common with R .

Consider first the case where Q has one element at least in common with R .

Here since Q has two elements in common with P and since P and R are parallel, it follows by Theorem 46 that Q has a second element in common with R .

Further, since Q and S are parallel and R has two elements in common with Q and has the element A_1 in common with S , it follows that R has a second element in common with S and therefore R and S have a general line, say c , in common.

Next consider the case where Q has no element in common with R .

This case has no analogue in ordinary three-dimensional Euclidean geometry, but must be considered in our system.

We have seen that there is at least one acceleration plane containing a and another general line, say b , in R since P and R are parallel.

Let T be such an acceleration plane, let A_2 be any element in b and let U be the acceleration plane through A_2 parallel to Q .

Then since Q and U are parallel and since T contains the general line a and also the element A_2 of U , it follows that T contains a general line, say b' , in U .

But the general lines b and b' both contain the element A_2 and neither of them can intersect a .

Thus since b and b' both lie in one acceleration plane T , it follows by Theorem 47 that they must be identical, and so b must be common to U and R .

Now the acceleration planes S and U are both parallel to Q and therefore must be either parallel to one another or else identical.

If they are not identical, the acceleration plane R has the general line b in common with U and has the element A_1 in common with S .

Thus in either case R and S have a general line in common.

If we consider case (2) of the last theorem it is clear that, if the general line a be an optical line, then since the general line b lies in the same acceleration plane T and has no element in common with a , it follows by Theorem 47 that b must also be an optical line and be parallel to a .

If c be the general line common to R and S then, provided c and b are distinct, it follows in a similar manner that c is an optical line parallel to b and therefore also parallel to a .

A similar result follows in case (1), and so we always have c parallel to a provided a be an optical line.

Now we have as yet given no definition of the parallelism of any type of general lines except optical lines, but are now in a position to do so.

Definition. If a be a general line and A be any element which does not lie in it and if two acceleration planes R and S through A are parallel respectively to two others P and Q containing a , then the general line which R and S have in common is said to be *parallel* to a .

THEOREM 49.

If a be a general line and A_1 be any element which does not lie in it, then there is one single general line containing A_1 and parallel to a .

Two cases have to be considered.

- (1) The element A_1 lies in an acceleration plane containing a .
- (2) The element A_1 does not lie in an acceleration plane containing a .

Consider first case (1) and let T be the acceleration plane containing A_1 and a .

Let P_1, P_2, P_3, P_4 be any other acceleration planes containing a , and let Q_1, Q_2, Q_3, Q_4 be acceleration planes through A_1 parallel to P_1, P_2, P_3, P_4 respectively.

Then since the acceleration plane T has the general line a in common with P_1 and has the element A_1 in common with Q_1 , it follows that it has a general line, say b , in common with Q_1 and b does not intersect a .

But, by Theorem 47, there is only one general line through A_1 in the acceleration plane T which does not intersect a and so b must be this general line.

Similarly Q_2, Q_3, Q_4 must all contain the general line b in common with T and so any pair of the acceleration planes Q_1, Q_2, Q_3, Q_4 have the same general line b in common.

Thus b is independent of the particular pair of acceleration planes P_1, P_2, P_3, P_4 which we may select and so there is only one general line through A_1 parallel to a .

Suppose next that A_1 does not lie in an acceleration plane containing a and suppose that P_1, P_2, P_3, P_4 are any acceleration planes which are distinct from one another and all contain a .

Let Q_1, Q_2, Q_3, Q_4 be acceleration planes through A_1 and parallel to P_1, P_2, P_3, P_4 respectively.

Let P_n be an acceleration plane containing a and a general line b in Q_1 .

Then b is parallel to a and lies in the same acceleration plane P_n with it.

If then we take acceleration planes Q'_2, Q'_3, Q'_4 through any element of b and parallel to P_2, P_3, P_4 respectively, these will all contain b and will also be respectively parallel to Q_2, Q_3, Q_4 which contain the element A_1 .

But the general line b and the element A_1 lie in the acceleration plane Q_1 and so, by case (1), Q_2, Q_3, Q_4 all have the same general line, say c , in common with Q_1 .

Thus any pair of the acceleration planes Q_1, Q_2, Q_3, Q_4 have the same general line c in common.

It follows that c is independent of the particular pair of the acceleration planes P_1, P_2, P_3, P_4 which we may select and so there is only one general line through A_1 parallel to a .

Thus the theorem holds in general.

THEOREM 50.

If two distinct general lines are each parallel to a third, then they are parallel to one another.

Let a and b be two distinct general lines which are each parallel to the general line c .

Let R_1 and R_2 be two acceleration planes each containing c but not containing a or b .

Let P_1 and P_2 be two acceleration planes parallel respectively to R_1 and R_2 and through any element of a .

Then P_1 and P_2 each contain a .

Similarly let Q_1 and Q_2 be two acceleration planes parallel respectively to R_1 and R_2 and containing b .

Then Q_1 is either parallel to P_1 or identical with it, while Q_2 is either parallel to P_2 or identical with it.

In either case we must have a parallel to b .

REMARKS.

If a and b be any pair of parallel general lines, it is easy to see that they must be general lines of the same kind, for we know already that two parallel general lines in one acceleration plane must be of the same kind and by two applications of this result it follows that if a and b do not lie in one acceleration plane they must also be of the same kind.

THEOREM 51.

If two parallel general lines a and b lie in one acceleration plane R and if two other distinct acceleration planes P and Q containing a and b respectively have an element A_1 in common, then P and Q have a general line in common which is parallel to a and b .

Let any element in b be selected and let S be the acceleration plane through this element and parallel to P .

Then the general line b must lie in S and so since Q contains the general line b and the element A_1 it follows that P and Q contain a general line in common which is parallel to b and therefore also parallel to a .

THEOREM 52.

If a pair of non-parallel general lines a and b lie in one acceleration plane P and if through an element A_1 not lying in the acceleration plane there are two other general lines c and d respectively parallel to a and b , then c and d lie in an acceleration plane parallel to P .

Let R be any acceleration plane distinct from P which contains a but not A_1 , and let S be any acceleration plane distinct from P which contains b but not A_1 .

Let P' be the acceleration plane through A_1 parallel to P , while R' and S' are the acceleration planes through A_1 parallel to R and S respectively.

Then P' and R' have a general line in common which is parallel to a and since it passes through A_1 must be identical with c ; while P' and S' have a general line in common which is parallel to b and since it passes through A_1 must be identical with d .

Thus c and d lie in the acceleration plane P' which is parallel to P .

THEOREM 53.

If three distinct acceleration planes P , Q and R and three parallel general lines a , b and c be such that a lies in P and R , b in Q and P and c in R and Q , then if Q' be an acceleration plane parallel to Q through some element of P which does not lie in b the acceleration planes R and Q' have a general line in common which is parallel to c .

Since the acceleration plane P contains two elements in common with Q and one element in common with the parallel acceleration plane Q' , it follows by Theorem 46 that P and Q' have two elements in common and therefore have a general line in common which is parallel to b . Call this general line d .

If this general line should happen to coincide with a , the result follows directly.

We shall therefore consider the case where it does not coincide with a .

Let A be any element in a .

Then, in case a be an optical line, the other optical line through A in the acceleration plane P will intersect b , while, if a be an inertia or separation line, both the optical lines through A in the acceleration plane P will intersect b .

Thus in all cases there is at least one optical line through A in the acceleration plane P which intersects b .

Let such an optical line intersect b in B and let an optical line through B in the acceleration plane Q intersect c in C .

Then BA and BC may be taken as generators of opposite sets of an acceleration plane, say S , which contains A , B and C .

Now the general line a is parallel to b and therefore also parallel to d , and, since BA passes through A , is distinct from a , and lies in the acceleration plane P , it follows that BA intersects d in some element, say D , which accordingly lies in the acceleration plane Q .

But since D lies in BA it lies in the acceleration plane S and thus S contains two elements (B and C) in common with Q and an element D in common with the parallel acceleration plane Q' .

It follows by Theorem 46 that S contains a second element in common with Q' and so S and Q' contain a general line in common which must be parallel to CB .

If we call this general line in S and Q' g , then any general line through C in the acceleration plane S , with the exception of CB , must intersect g .

But the element A does not lie in b and so does not lie in the acceleration plane Q and therefore does not lie in CB .

Thus since the general line CA is distinct from CB , and since CA must lie in S , it follows that CA must intersect g in some element, say F .

But C and A both lie in the acceleration plane R which accordingly must contain the general line CA and therefore the element F .

Thus since the acceleration plane R contains the general line c in common with Q and contains the element F in the parallel acceleration plane Q' , it follows that R must have a general line in common with Q' and this general line must be parallel to c .

THEOREM 54.

(a) *If a and b be two parallel separation lines in the same acceleration plane and if one element of b be before an element of a , then each element of b is before an element of a .*

Let A be the element of b which by hypothesis is *before* an element of a .

Let the two optical lines through A in the acceleration plane be called c and d .

Let B be any other element of b .

Then by Theorem 40 B must be *before* an element of one of the optical lines c and d and *after* an element of the other.

It will be sufficient to consider the case when B is *before* an element of c and *after* an element of d , since the proof in the other case is similar.

Let e and f be the two optical lines through B in the acceleration plane and let e be the one which is parallel to c . Then f intersects c in some element C .

Also c intersects a in some element D (Post. XV) and D must be *after* A ; for since A is *before* an element of a , we should otherwise have one element of a *after* another, contrary to the hypothesis that a is a separation line.

Now since B is *before* an element of c and cannot also be *after* an element of c , and since C lies in the optical line f through B , it follows that C is *after* B .

Now C cannot be *before* A for then A would be *after* B , contrary to the hypothesis that A and B lie in a separation line.

If C be either *before* D or coincident with D , then B is *before* D an element of a .

Suppose next that C is *after* D and let E be the element in which f intersects a .

Let h be the second optical line through D in the acceleration plane and let g be the second optical line through E in the acceleration plane and let g and h intersect in F .

Then the optical lines c , f , h and g form an optical parallelogram whose diagonal line through D and E is a .

Let j be the other diagonal line through c and f ; then j is an inertia line.

Let the optical lines d and e intersect in G .

Then the optical lines c , f , d and e form an optical parallelogram whose diagonal line through A and B is b .

Thus in the two optical parallelograms since the diagonal lines a and b do not intersect, it follows that the diagonal lines of the other kind do not intersect (Post. XVI).

But the two optical parallelograms have the corner C in common and so they must have a diagonal line in common and so G must lie in j .

Also D is *after* A and so h must be an after-parallel of d .

But since F and G are elements of j which is an inertia line, it follows that the one is *after* the other; and since no element of d can be *after* an element of h , it follows that F must be *after* G .

Thus since F is an element of g and G is an element of e , it follows that g is an after-parallel of e .

But since E and B lie in the optical line f , one of them must be *after* the other, and since B lies in e it cannot be *after* E which is an element of g .

Thus E is *after* B and so B is *before* an element of a .

Thus in all cases B is *before* an element of a .

(b) *If a and b be two parallel separation lines in the same acceleration plane and if one element of b be after an element of a , then each element of b is after an element of a .*

THEOREM 55.

(a) *If a and b be a pair of parallel separation lines in the same acceleration plane and if an optical line c intersects a in A_1 and b in B_1 while a parallel optical line d intersects a in A_2 and b in B_2 , then if B_1 is before A_1 we have also B_2 before A_2 .*

By Theorem 54, since B_1 is *before* A_1 , therefore B_2 is *before* an element of a .

But since A_2 and B_2 are distinct elements in the optical line d , therefore one of them is *after* the other.

Further B_2 could not be *after* A_2 for then since B_2 is *before* an element of a we should have A_2 *before* this element of a , contrary to the hypothesis that a is a separation line.

Thus B_2 must be *before* A_2 .

(b) *If a and b be a pair of parallel separation lines in the same acceleration plane and if an optical line c intersects a in A_1 and b in B_1 while a parallel optical line d intersects a in A_2 and b in B_2 , then if B_1 is after A_1 we have also B_2 after A_2 .*

THEOREM 56.

(a) *If a and b be a pair of parallel inertia lines in the same acceleration plane and if an optical line c intersect a in A_1 and b in B_1 , while a parallel optical line d intersects a in A_2 and b in B_2 ; then if B_1 is before A_1 we have also B_2 before A_2 .*

Since B_1 and B_2 are elements of an inertia line b , one of them must be *after* the other.

We shall first consider the case when B_2 is *after* B_1 .

Let e be the second optical line through B_2 in the acceleration plane.

Then since by hypothesis d is parallel to c , it follows that e must intersect c in some element C .

Then C must be *after* B_1 , for if C were *before* B_1 , then B_1 would lie in the α sub-set of C and by Post. XIV (a) B_1 would be the only element common to the inertia line b and the α sub-set of C .

Also by Post. XIV (b) there would be *one single element* common to the inertia line b and the β sub-set of C and since there are only two optical lines through C in the acceleration plane this element would have to be identical with B_2 .

Thus we should have B_2 *before* C and C *before* B_1 and therefore B_2 *before* B_1 , contrary to the hypothesis that B_2 is *after* B_1 .

Thus we see that C must be *after* B_1 and since thus B_1 must be in the β sub-set of C , it follows by Post. XIV (a) that there is *one single element* common to the inertia line b and the α sub-set of C .

Since there are only two optical lines through C in the acceleration plane, it follows that this element must be identical with B_2 .

Let the optical line e intersect a in D .

If then C is *before* A_1 we shall have A_1 in the α sub-set of C and by Post. XIV (b) there is *one single element* common to the inertia line a and the β sub-set of C , and since there are only two optical lines through C in the acceleration plane, it follows that this element must be identical with D .

Thus D is *before* C and C is *before* B_2 and consequently D is *before* B_2 and since D and B_2 lie in one optical line it follows that D lies in the β sub-set of B_2 .

If C were identical with A_1 , it would also be identical with D and again D would lie in the β sub-set of B_2 .

But by Post. XIV (a) there is *one single element* common to the inertia line a and the α sub-set of B_2 and since there are only two optical lines through B_2 in the acceleration plane this element must lie in d and must therefore be identical with A_2 .

Thus since A_2 lies in the α sub-set of B_2 and is not identical with B_2 , therefore B_2 must be *before* A_2 .

Thus in case C is either *before* A_1 or identical with A_1 we have B_2 *before* A_2 .

Next suppose C is *after* A_1 and let f be the second optical line through B_1 in the acceleration plane.

Let f and d intersect in the element F .

Then the optical lines e, d, c and f form an optical parallelogram whose diagonal line through B_1 and B_2 is b .

Let j be the other diagonal line through C and F .

Then since b is an inertia line, j must be a separation line.

Again let g be the second optical line through D in the acceleration plane and let h be the second optical line through A_1 in the acceleration plane and let g and h intersect in E .

Then the optical lines e, g, c and h form an optical parallelogram whose diagonal line through A_1 and D is a .

Thus the two optical parallelograms formed by e, d, c and f and by e, g, c and h have diagonal lines of one kind, b and a , which do not intersect and so by Post. XVI their diagonal lines of the other kind do not intersect.

But the two optical parallelograms have the corner C in common and so they have the diagonal line through C in common.

Thus E lies in j and since j is a separation line E is neither *before* nor *after* F .

But since A_1 is *after* B_1 it follows that h is an after-parallel of f and so E must be *after* an element of f .

But since E is neither *before* nor *after* F it follows by Theorem 40 that since E is *after* an element of f it must be *before* an element of d .

Thus g is a before-parallel of d and since D and B_2 lie in the optical line e which intersects g in D and d in B_2 , it follows that D is *before* B_2 .

Thus D lies in the β sub-set of B_2 and in the optical line e .

But by Post. XIV (a) there is *one single element* common to the inertia line a and the α sub-set of B_2 and since there are only two optical lines through B_2 in the acceleration plane it follows that this element must lie in d and is therefore identical with A_2 .

Thus since A_2 is in the α sub-set of B_2 and is not identical with B_2 , therefore B_2 is *before* A_2 .

This proves the theorem provided B_2 is *after* B_1 .

Suppose now that B_1 is *after* B_2 .

Then A_1 must be *after* A_2 for in the first place it cannot be identical with it since the optical lines c and d are parallel and so cannot have an element in common.

Further, since the element B_1 is *after* B_2 , it follows that c must be an after-parallel of d and so A_1 must be *after* an element of the optical line d .

But since A_1 and A_2 are distinct elements of the inertia line a , the one must be *after* the other, and since, by Theorem 12, A_1 cannot be *before* one element of the optical line d and *after* another element of it without lying in the optical line, it follows that the element A_1 must be *after* the element A_2 .

Suppose now, if possible, that A_2 is *before* B_2 , then reversing the rôles of the inertia lines a and b it would follow from what we have already proved that, c and d being parallel, A_1 would have to be *before* B_1 , contrary to hypothesis.

Thus since B_2 must be either *after* or *before* A_2 and cannot be *after*, it follows that B_2 is *before* A_2 .

(b) *If a and b be a pair of parallel inertia lines in the same acceleration plane and if an optical line c intersect a in A_1 and b in B_1 , while a parallel optical line d intersects a in A_2 and b in B_2 ; then if B_1 is after A_1 we have also B_2 after A_2 .*

Since a pair of parallel inertia lines always lie in an acceleration plane, the words "in the same acceleration plane" may be omitted in the enunciation of this theorem.

THEOREM 57.

If two elements A and B lie in one optical line and if two other elements C and D lie in a parallel optical line in the same acceleration plane, then if A be after B and C after D the general lines AD and BC intersect.

Let a be the optical line containing A and B , and let b be the parallel optical line containing C and D .

Then the general lines AD and BC cannot be parallel optical lines, for since B is *before* A an optical line through B which intersected b would be a before-parallel of an optical line through A which intersected b and so the element in which the former optical line intersected b would be *before* the element in which the latter optical line intersected b .

Further, Theorems 55 and 56 show that AD and BC cannot be either parallel separation lines or parallel inertia lines.

Again AD and BC cannot both be optical lines for we know that two optical lines which intersect a pair of parallel optical lines are themselves parallel.

Thus we are left with the following possibilities as to the general lines AD and BC .

- (1) One is an optical line and the other an inertia line.
- (2) One is an optical line and the other a separation line.
- (3) One is a separation line and the other an inertia line.
- (4) Both are inertia lines.
- (5) Both are separation lines.

In case (1) Theorem 36 shows that the general lines intersect.

In cases (2) and (3) it follows from Post. XV that the general lines intersect.

In cases (4) and (5), since we have shown that the two general lines cannot be parallel, it follows by Theorem 47 that they must intersect.

Thus in all cases the general lines AD and BC intersect.

Definitions. If four optical lines form an optical parallelogram, they will be spoken of as the *side lines* of the optical parallelogram.

A pair of side lines which do not intersect will be called *opposite*.

The element of intersection of the diagonal lines will be spoken of as the *centre* of the optical parallelogram.

THEOREM 58.

If any two distinct elements A and O be taken in an inertia or separation line i in a given acceleration plane, then there is one single optical parallelogram in the acceleration plane having O as the centre and A as one of its corners.

Let a and b be the two optical lines through A in the acceleration plane while c and d are the ones through O ; the optical line c being parallel to a and the optical line d parallel to b .

Let j be the second diagonal line of the optical parallelogram formed by a , b , c and d .

Then by Theorem 47 there is one single general line through O and parallel to j .

Call this general line k and let a intersect k in D while b intersects k in C .

The elements of intersection must exist since k , being parallel to j , must be an inertia or separation line according as i is a separation or inertia line; while a and b are both optical lines.

Let e be the second optical line through C in the acceleration plane, while f is the second optical line through D in the acceleration plane, and let e and f intersect in B .

Then a , b , e and f form an optical parallelogram in the same acceleration plane with that formed by a , b , c and d and their diagonal lines of one kind k and j do not intersect and so by Post. XVI their diagonal lines of the other kind do not intersect.

But the corner A is common to both optical parallelograms and so the diagonal line i which passes through that corner must be a diagonal line of both optical parallelograms.

Thus B must lie in i and so O is the centre of the optical parallelogram formed by a , b , e and f , while A is one of its corners.

Again, if there were a second optical parallelogram in the acceleration plane having O as centre and A one of its corners, then such an optical parallelogram would have i as one of its diagonal lines and so the other diagonal lines of the two optical parallelograms would not intersect.

Further since the two optical parallelograms have the element O common to these other diagonal lines, the latter must be identical.

But there are only two optical lines, a and b , through A in the acceleration plane and these intersect k in D and C respectively, which must accordingly be a pair of opposite corners of the second optical parallelogram.

But then the second optical parallelogram would have e and f as its remaining side lines and so could not be distinct from the first optical parallelogram.

Thus there is no second optical parallelogram in the acceleration plane having O as centre and A as one of its corners.

THEOREM 59.

If two optical parallelograms have two opposite corners in common, then they have a common centre.

Two cases are possible :

- (1) The common opposite corners may lie in an inertia line.
- (2) The common opposite corners may lie in a separation line.

We shall consider first the case where they lie in an inertia line.

Let A and B be the two common opposite corners of the optical parallelograms: B being *after* A .

Let C and D be the other pair of opposite corners of the one optical parallelogram which we shall suppose to lie in an acceleration plane P , while C' and D' are the other pair of opposite corners of the other

optical parallelogram which we shall suppose to lie in an acceleration plane P' .

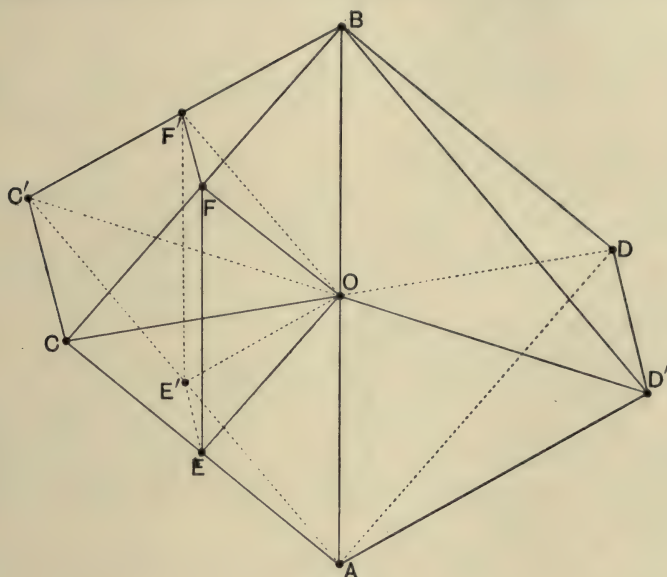


Fig. 9.

Then P and P' must be distinct if the optical parallelograms are distinct.

Let O be the centre of the optical parallelogram whose corners are A, B, C, D and let OE and OF be optical lines through O parallel to CB and AC respectively and intersecting AC and CB in E and F respectively.

Then E, C, F and O form the corners of an optical parallelogram in the acceleration plane P , and this optical parallelogram and the one whose corners are A, C, B and D have the common diagonal line CD and so their diagonal lines of the other kind do not intersect.

Thus AB and EF are parallel and EF is an inertia line.

Now let OE' and OF' be optical lines through O parallel to $C'B$ and AC' respectively and intersecting AC' and $C'B$ in E' and F' respectively.

Then AC and AC' may be taken as generators of opposite sets of an acceleration plane Q_1 , while OF and OF' will be generators of opposite sets of a parallel acceleration plane Q_2 .

Similarly BC and BC' may be taken as generators of opposite sets of an acceleration plane R_1 , while OE and OE' will be generators of opposite sets of a parallel acceleration plane R_2 .

But Q_1 and R_1 have the general line CC' in common while Q_1 and R_2 have the general line EE' in common and so since R_1 and R_2 are parallel it follows that CC' and EE' are parallel.

Again since R_1 and Q_2 have the general line FF' in common and since Q_1 and Q_2 are parallel, it follows that FF' and CC' are parallel.

Thus FF' is parallel to EE' .

But since EF is an inertia line there exists an acceleration plane containing E , F and F' . Let S be this acceleration plane.

Then there exists in S a general line through E which is parallel to FF' and, since there can be only one parallel to FF' through E , this must be identical with the general line EE' .

Thus E' must lie in the acceleration plane S .

But since AB and EF are parallel and lie in P while P' and S are two other distinct acceleration planes containing AB and EF respectively and since P' and S have an element F' in common, it follows by Theorem 51 that the general line $E'F'$ which is common to P' and S is parallel to AB .

But now E' , C' , F' and O form the corners of an optical parallelogram in the acceleration plane P' , and this optical parallelogram and the one whose corners are A , C' , B and D' have one pair of diagonal lines, namely $E'F'$ and AB , which do not intersect and so their diagonal lines of the other kind do not intersect.

But these latter diagonal lines are $C'O$ and $C'D'$ respectively and so since they have the element C' in common it follows that they are identical.

Thus the element O must lie in $C'D'$ and since it also lies in AB it follows that O is the centre of the optical parallelogram whose corners are A , C' , B , D' .

Thus the optical parallelograms having A and B as opposite corners have a common centre O .

We have next to consider the case where the common opposite corners lie in a separation line.

Let A and B be the two common opposite corners of the optical parallelograms: B being neither *before* nor *after* A .

Let C and D be the other pair of opposite corners of the one optical parallelogram, which we shall suppose to lie in an acceleration plane P , while C' and D' are the other pair of opposite corners of the other optical parallelogram, which we shall suppose to lie in an acceleration plane P' .

Then P and P' must be distinct if the optical parallelograms are distinct.

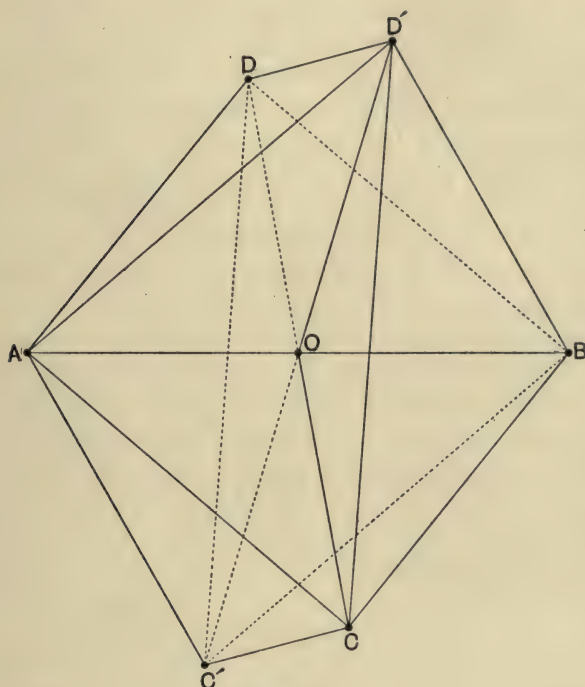


Fig. 10.

We shall further suppose D to be *after* C and D' *after* C' .

Now the following pairs of intersecting optical lines may be taken as generators of opposite sets of certain acceleration planes which we shall denote by the following symbols opposite each pair.

| Optical lines | Acceleration plane |
|----------------|--------------------|
| CA and $C'A$ | Q_1 |
| BD and BD' | Q_2 |
| CB and $C'B$ | R_1 |
| AD and AD' | R_2 |
| AC' and AD | S_1 |
| BD' and BC | S_2 |
| BC' and BD | T_1 |
| AD' and AC | T_2 |

Of these acceleration planes we evidently have those pairs parallel which are represented by the same letters.

Thus the general line $C'D$, since it lies in S_1 and T_1 , must be parallel to the general line CD' , since the latter lies in S_2 and T_2 .

Similarly the general line DD' , since it lies in Q_2 and R_2 , must be parallel to the general line $C'C$, since the latter lies in Q_1 and R_1 .

But CD is an inertia line and so there is an acceleration plane containing C , D and D' , and if we call this acceleration plane U then U contains the general lines CD' and DD' and so U must also contain the general lines through D parallel to CD' and through C parallel to DD' .

That is: the acceleration plane U must contain $C'D$ and $C'C$.

Thus U must contain C' and therefore contains $C'D'$.

Thus the centres of the two optical parallelograms must lie in the acceleration plane U and in the separation line AB .

The acceleration plane U cannot however have more than one element in common with AB , for otherwise it would contain both A and B , and since U contains D we should have U identical with P ; but U contains D' which does not lie in P and so this is impossible.

Thus the element in which CD intersects AB must be identical with the element in which $C'D'$ intersects AB , or in other words the two optical parallelograms have a common centre.

Thus the theorem is proved.

THEOREM 60.

If two optical parallelograms have two adjacent corners in common, then optical lines through the centres of the optical parallelograms and intersecting their common side line intersect it in the same element.

Let A and B be the two common adjacent corners of two optical parallelograms which we shall suppose to lie in separate acceleration planes P and P' .

We shall suppose C and D to be the other corners of the optical parallelogram in P and shall suppose C to be opposite to B and D opposite to A .

We may further, without limitation of generality, take the diagonal line CB as the inertia diagonal line.

We shall suppose C' and D' to be the remaining corners of the optical parallelogram in P' and we shall take C' opposite to B and D' opposite to A .

Let O be the centre of the optical parallelogram in P and let the one optical line through O in the acceleration plane P intersect AB in M , while the other optical line in P through O intersects AC in E .

Then A , E , O and M form the corners of an optical parallelogram also in the acceleration plane P .

The optical parallelograms whose corners are A, E, O, M and A, C, D, B have the diagonal line AD in common and so, by Post. XVI, their diagonal lines of the other kind do not intersect.

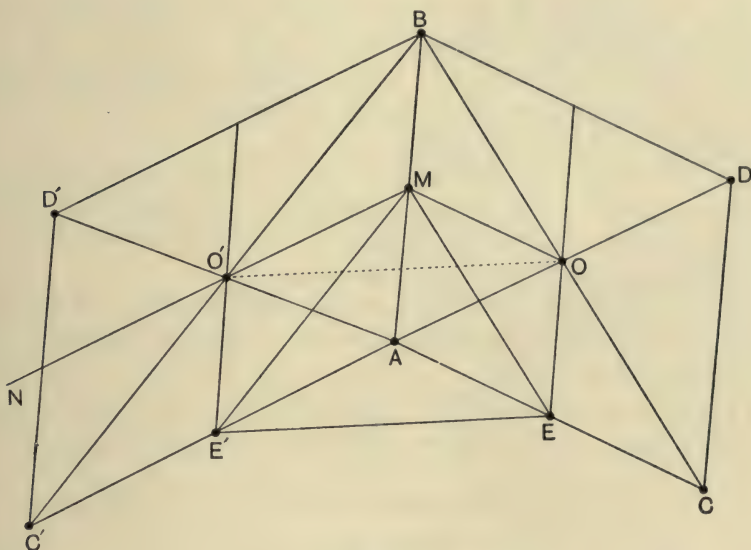


Fig. 11.

Thus EM and CB are parallel.

Now let MN be the optical line through M parallel to AC' and let MN intersect the diagonal line $C'B$ in O' .

Let $O'E'$ be the optical line through O' parallel to MA and intersecting AC' in E' .

Then $O'E'$ is parallel to OE and unless it be a neutral-parallel we have $O'E'$ and OE in one acceleration plane.

Now since MN is an optical line through M which neither intersects OE nor is parallel to it, it follows by Post. XII that there is *one single element* in MN which is neither *before* nor *after* any element of OE .

If O_0 be this element, we shall suppose first that O' is distinct from O_0 and thereby ensure that $O'E'$ and OE lie in one acceleration plane.

Call this acceleration plane Q .

Now since MO and MO' are respectively parallel to AE and AE' and all four are optical lines, it follows that M, O and O' lie in one acceleration plane, say R_1 , while A, E and E' lie in a parallel acceleration plane, say R_2 .

But Q has the elements O and O' in common with R_1 and has the

elements E and E' in common with R_2 and so the general lines OO' and EE' are parallel.

We have however further seen that OB and EM are parallel and are both inertia lines.

Thus O , O' and B lie in one acceleration plane, say S_1 , while E , E' and M lie in a parallel acceleration plane, say S_2 .

But the acceleration plane P' has the elements O' and B in common with S_1 and has the elements E' and M in common with S_2 .

Thus BO' and ME' are parallel.

But BO' is the same general line as BC' , which is a diagonal line of the optical parallelogram whose corners are A , C' , D' , B , while ME' is a diagonal line of the optical parallelogram whose corners are A , E' , O' , M and these diagonal lines do not intersect.

It follows by Post. XVI that their other diagonal lines AD' and AO' do not intersect and so since they have the element A in common they must be identical.

Thus O' must lie in AD' and since it also lies in BC' , it follows that O' is the centre of the optical parallelogram whose corners are A , C' , D' , B .

Thus the optical lines through the centres O and O' and intersecting AB , intersect it in the same element M .

Now this same method of proof holds for the case of any optical parallelogram in the acceleration plane P' which has A and B as adjacent corners, provided that the diagonal line through B does not intersect MN in O_0 , and so all such optical parallelograms have their centres in the optical line MN .

Again, if we select a second optical parallelogram in the acceleration plane P having A and B as adjacent corners but not having O as centre, we may use a similar method of proof and show that all optical parallelograms in the acceleration plane P' having A and B as adjacent corners have, with *one possible exception*, got their centres in one optical line.

This *one possible exception* is however different from the *one possible exception* which we found before and so it follows that no exception exists.

Similar considerations show that all optical parallelograms in the acceleration plane P having A and B as adjacent corners, have their centres in one optical line MO .

Thus the theorem holds for optical parallelograms in the acceleration planes P and P' and will therefore also hold for optical parallelograms in any other acceleration planes which contain A and B .

Definition. If A and B be two distinct elements lying in an inertia line or in a separation line, then the centre of an optical parallelogram of which A and B are a pair of opposite corners will be spoken of as the *mean* of the elements A and B .

Theorem 59 shows that if two elements A and B lie in an inertia or separation line their mean is independent of the particular optical parallelogram used to define it.

Since a diagonal line of an optical parallelogram is either an inertia or a separation line, the above definition fails for the case of two distinct elements lying in an optical line.

In this case we adopt the following definition.

Definition. If A and B be two distinct elements lying in an optical line, then an optical line through the centre of an optical parallelogram of which A and B are a pair of adjacent corners and intersecting the optical line AB , intersects it in an element which will be spoken of as the *mean* of the elements A and B .

Theorem 60 shows that if two elements A and B lie in an optical line, their mean is independent of the particular optical parallelogram used to define it.

THEOREM 61.

If two or more optical parallelograms have a pair of opposite side lines in common, their centres lie in a parallel optical line in the same acceleration plane.

We have already seen in the course of proving the last theorem that this result must hold if the two optical parallelograms have a third side in common.

In case this is not so, let A_1, B_1, C_1, D_1 be four distinct elements in an optical line a and let b be a parallel optical line in an acceleration plane containing a .

Let the second optical lines through A_1, B_1, C_1, D_1 respectively in the acceleration plane intersect b in A_2, B_2, C_2, D_2 respectively and let A_1, B_1, A_2, B_2 be the corners of one of the optical parallelograms under consideration and C_1, D_1, C_2, D_2 the corners of another.

Then A_1, D_1, A_2, D_2 is a third optical parallelogram.

Call these optical parallelograms (1), (2) and (3) and let their centres be O, O', O'' respectively.

Then by the first case O and O'' lie in an optical line parallel to a and b since (1) and (3) have the pair of adjacent corners A_1 and A_2 in common.

Similarly O' and O'' lie in an optical line parallel to a and b since (2) and (3) have the pair of adjacent corners D_1 and D_2 in common.

But there is only one optical line through O'' parallel to a and b and so O , O' and O'' lie in one optical line parallel to a and b .

Thus all optical parallelograms having a and b as a pair of opposite side lines must have their centres in the optical line OO' .

THEOREM 62.

If two optical parallelograms have a pair of opposite side lines in common and if one diagonal line of the one optical parallelogram passes through the centre of the other, then the two optical parallelograms have a common centre.

Since the centre of an optical parallelogram is the element of intersection of its diagonal lines, and since by hypothesis one diagonal line of the one optical parallelogram passes through the centre of the other, it follows that both centres must lie in that diagonal line.

Now we know that in any optical parallelogram the one diagonal line is an inertia line, while the other is a separation line.

Thus the centres of the two optical parallelograms must lie in an inertia line or a separation line.

But we have already seen by Theorem 61 that they lie in an optical line, and since any two distinct elements determine a general line, it follows that the centres cannot be distinct.

Thus the two optical parallelograms have a common centre.

THEOREM 63.

If two optical parallelograms P and Q in the same acceleration plane have a common centre, then the elements in which a pair of opposite side lines of P intersect the diagonal lines of Q form the corners of an optical parallelogram with the same centre.

Let O be the common centre of the two optical parallelograms P and Q and let i and j be the two diagonal lines of Q while a and b are a pair of opposite side lines of P .

Let a intersect i in E and j in F , while b intersects i in G and j in H .

Denote the second optical line through E in the acceleration plane by c and suppose it intersects b in H' .

Denote the second optical line through G in the acceleration plane by d , and suppose it intersects a in F' .

Then the optical lines a , c , b and d form an optical parallelogram one of whose diagonal lines, namely i , passes through O the centre

of the optical parallelogram P of which a and b are opposite side lines, and so by Theorem 62 these two optical parallelograms have a common centre O .

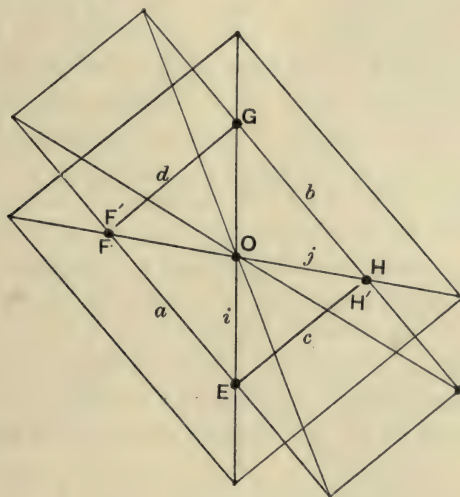


Fig. 12.

Thus if j' be the second diagonal line of the optical parallelogram formed by a, c, b and d , it has the element O in common with j .

The two optical parallelograms Q and that formed by a, c, b and d have however the diagonal line i in common and thus their diagonal lines of one kind do not intersect and so by Post. XVI their diagonal lines of the other kind do not intersect.

But these diagonal lines are j and j' which as we have seen have the element O in common and therefore must be identical.

Thus F' must be identical with F and H' must be identical with H and so the elements E, F, G and H must form the corners of an optical parallelogram having the same centre as the two original optical parallelograms, as was to be proved.

REMARKS AND DEFINITIONS.

If a and b be any two distinct inertia lines and A_0 be any element in a which is not an element of intersection with b , then from Post. XIV (a) it follows that there is one single element common to the inertia line b and the α sub-set of A_0 .

Call this element B_0 .

Then B_0 is distinct from A_0 and cannot be an element of intersection

of the two inertia lines, for if it were A_0 and B_0 would lie both in an inertia line and an optical line, which is impossible.

Further there cannot be an element of intersection of the two inertia lines lying *after* A_0 and *before* B_0 for by Theorem 1 (a) A_0 is *before* B_0 but is not *before* any element outside the α sub-set of A_0 which is *before* B_0 , and an element of intersection of the two inertia lines lying *after* A_0 and *before* B_0 would have this character.

Thus any element of intersection of the two inertia lines, if such an element exists, must lie either *before* A_0 or *after* B_0 .

Again from Post. XIV (a) it follows that there is one single element, say A_1 , common to the inertia line a and the α sub-set of B_0 , and again A_1 cannot be an element of intersection of the inertia lines.

Further any such element, if it exists, must lie either *before* A_0 or *after* A_1 .

Proceeding again in the same way there is one single element, say B_1 , common to the inertia line b and the α sub-set of A_1 and one single element A_2 common to the inertia line a and the α sub-set of B_1 and so on.

Thus we get an infinite series of elements $A_0, A_1, A_2, A_3, \dots$ in the inertia line a and another infinite series of elements $B_0, B_1, B_2, B_3, \dots$ in the inertia line b .

An element of intersection of the two inertia lines if such an element exists must lie either *before* A_0 or *after* A_n , where n is any finite integer whatever.

This process will be spoken of as *taking steps along the inertia line a with respect to the inertia line b*.

The passing from A_0 to A_1 is the first step, the passing from A_1 to A_2 the second, and so on.

If X be an element which is *after* A_0 in the inertia line a and *before* A_n but not *before* A_{n-1} , then the element X will be said to be *surpassed* from A_0 in n steps taken with respect to b .

If C be an element of intersection of the two inertia lines and if C be *after* A_0 , it is evident from what we have said that C cannot be *surpassed* from A_0 in any finite number of steps.

These remarks and definitions prepare the way for Postulate XVII.

POSTULATE XVII. If A_0 and A_x be two elements of an inertia line a such that A_x is *after* A_0 , and if b be a second inertia line which does not intersect a either in A_0 , A_x or any element both *after* A_0 and *before* A_x , then A_x may be *surpassed* in a finite number of steps taken from A_0 along a with respect to b .

This postulate will be found to take the place of the well-known axiom of Archimedes, to which it will be seen to bear a certain resemblance.

It, however, unlike the axiom of Archimedes, contains no reference to congruence.

It follows directly from Post. XVII that if the two inertia lines a and b do not intersect at all then A_x may always be surpassed in a finite number of steps.

If A_x lies before A_0 a similar result holds, but this may be proved without further assumptions and forms the subject of our next theorem.

THEOREM 64.

If A_0 and A_x be two elements of an inertia line a such that A_x is before A_0 , and if b be a second inertia line which does not intersect a either in A_0 , A_x , or any element both before A_0 and after A_x , then A_0 may be reached in a finite number of steps taken along a from an element before A_x in a and with respect to b .

By Post. XVII since A_0 is after A_x it follows that A_0 may be surpassed in a finite number of steps, say n , taken from A_x along a with respect to b .

Let the elements marking these steps in a be denoted by A_{x+1} , A_{x+2} , A_{x+3} , ... A_{x+n} and let the elements in b lying in the β sub-sets of these be denoted by B_x , B_{x+1} , B_{x+2} , ... B_{x+n-1} respectively.

Then A_0 may either coincide with A_{x+n-1} or be after it.

If A_0 coincides with A_{x+n-1} , then it is reached in $n-1$ steps taken along a from A_x .

Now there is one single element, say B_{x-1} , common to the inertia line b and the β sub-set of A_x and also one single element, say A_{x-1} , common to the inertia line a and the β sub-set of B_{x-1} .

Then A_{x-1} is before A_x and A_0 is reached in n steps taken along a from A_{x-1} with respect to b .

This proves the theorem if A_0 coincides with A_{x+n-1} .

Suppose next that A_0 does not coincide with A_{x+n-1} .

Then A_0 is after A_{x+n-1} and before A_{x+n} .

Let B_{-1} be the one single element common to the inertia line b and the β sub-set of A_0 and let A_{-1} be the one single element common to the inertia line a and the β sub-set of B_{-1} .

Let B_{-2} be the one single element common to the inertia line b and the β sub-set of A_{-1} and let A_{-2} be the one single element common to the inertia line a and the β sub-set of B_{-2} , and so on, till we get to an element A_{-n} .

Now B_{-1} cannot coincide with B_{x+n-1} for then A_0 and A_{x+n} would be two distinct elements of the inertia line a both lying in the α sub-set of B_{-1} , contrary to Post. XIV (a).

Further B_{-1} cannot be *after* B_{x+n-1} , for A_{x+n} is *after* A_0 and A_0 is *after* B_{-1} and therefore A_{x+n} is *after* B_{-1} .

But B_{x+n-1} lies in the β sub-set of A_{x+n} and is distinct from it, and therefore A_{x+n} is *after* B_{x+n-1} but is not *after* any element outside the sub-set which is *after* B_{x+n-1} .

Thus B_{-1} cannot be *after* B_{x+n-1} .

It follows that B_{-1} must be *before* B_{x+n-1} .

Similarly B_{x+n-2} must be *before* B_{-1} .

Also

| | |
|-------------|-------------------------------------|
| A_{-1} | must be <i>before</i> A_{x+n-1} , |
| A_{x+n-2} | must be <i>before</i> A_{-1} , |
| B_{-2} | must be <i>before</i> B_{x+n-2} , |
| B_{x+n-3} | must be <i>before</i> B_{-2} , |
| A_{-2} | must be <i>before</i> A_{x+n-2} , |
| A_{x+n-3} | must be <i>before</i> A_{-2} , |
| | |
| | |
| A_x | must be <i>before</i> A_{-n+1} , |
| A_{-n} | must be <i>before</i> A_x . |

Thus A_{-n} is an element in a which is *before* A_x , and A_0 may be reached in a finite number of steps taken from A_{-n} with respect to b along a .

Thus the theorem holds in general.

THEOREM 65.

(a) *If A_0 and A_x be two elements in an inertia line a which lies in the same acceleration plane with another inertia line b which does not intersect a in A_0 , A_x , or any element after the one and before the other, and if an optical line through A_0 intersects b in B_0 so that B_0 is after A_0 , then a parallel optical line through A_x will intersect b in an element which is after A_x .*

We shall first suppose that A_x is *after* A_0 .

Let the optical line through A_x parallel to A_0B_0 intersect b in B_x .

Then by Post. XVII A_x may be surpassed in a finite number of steps, say n , taken from A_0 along a with respect to b .

Let the elements (including A_0) marking these steps in a be $A_0, A_1, A_2, \dots A_n$ and let the elements in b lying in the α sub-sets of these be $B_0, B_1, B_2, \dots B_n$ respectively.

Then A_x may either coincide with A_{n-1} or be *after* it.

Now the optical line B_0A_1 intersects the two optical lines A_0B_0 and A_1B_1 and so these latter two optical lines belong to one set and are therefore parallel.

Similarly A_1B_1 intersects the two optical lines B_0A_1 and B_1A_2 and so these two are also parallel but belong to the other set.

Proceeding thus we see that the optical lines A_0B_0 , A_1B_1 , A_2B_2 , ... A_nB_n belong to one set and are all parallel, while B_0A_1 , B_1A_2 , B_2A_3 , ... $B_{n-1}A_n$ belong to the other set and are all parallel.

| | | |
|-----|-----------|---|
| But | A_1 | lies in the α sub-set of B_0 , |
| | B_1 | " " " " A_1 , |
| | A_2 | " " " " B_1 , |
| | B_{n-1} | " " " " A_{n-1} , |
| | A_n | " " " " B_{n-1} , |
| | B_n | " " " " A_n . |

Thus if A_x coincides with A_{n-1} , then B_x must coincide with B_{n-1} and therefore B_x must lie in the α sub-set of A_x and since B_x and A_x are distinct it follows that B_x is *after* A_x and the optical lines A_0B_0 and A_xB_x are parallel.

This proves the theorem in this case.

If A_x does not coincide with A_{n-1} , then it must be *after* A_{n-1} and *before* A_n .

Also since A_xB_x is parallel to A_0B_0 it must be parallel to $A_{n-1}B_{n-1}$ and to A_nB_n .

But since A_x is *after* A_{n-1} and *before* A_n it follows that A_xB_x is an after-parallel of $A_{n-1}B_{n-1}$ and a before-parallel of A_nB_n .

Further A_xB_x must intersect the optical line $B_{n-1}A_n$ in some element, say C , since $B_{n-1}A_n$ is an optical line of the opposite set to A_xB_x and so C must be *after* B_{n-1} and *before* A_n .

Thus B_{n-1} must lie in the β sub-set of C , while A_n lies in the α sub-set of C .

But by Post. XIV (a) there is one single element common to the inertia line b and the α sub-set of C and this must lie in the other optical line through C in the acceleration plane; that is to say in the optical line A_xB_x and must therefore be identical with B_x .

Similarly by Post. XIV (b) there is one single element common to the inertia line a and the β sub-set of C and this must be identical with A_x .

Thus C is *after* A_x and *before* B_x and therefore B_x is *after* A_x .

Thus the theorem is proved for all cases in which A_x is *after* A_0 .

A similar method shows that the theorem is true when A_x is *before* A_0 except that Theorem 64 takes the place of Post. XVII.

Thus the theorem holds in general.

(b) If A_0 and A_x be two elements in an inertia line a which lies in the same acceleration plane with another inertia line b which does not intersect a in A_0 , A_x , or any element before the one and after the other, and if an optical line through A_0 intersects b in B_0 so that B_0 is before A_0 , then a parallel optical line through A_x will intersect b in an element which is before A_x .

THEOREM 66.

(a) If A_0 and A_x be two elements in a separation line a which lies in the same acceleration plane with another separation line b which does not intersect a in A_0 , A_x or any element lying between a pair of parallel optical lines through A_0 and A_x in the acceleration plane, and if an optical line through A_0 intersects b in B_0 so that B_0 is after A_0 , then a parallel optical line through A_x will intersect b in an element which is after A_x .

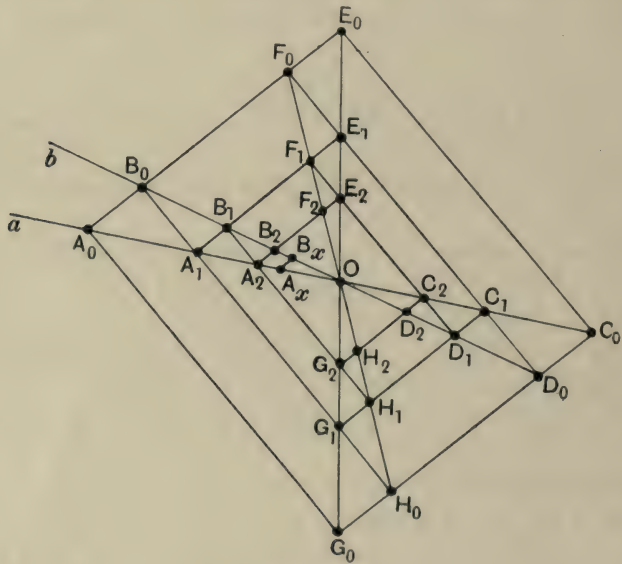


Fig. 13.

In case the separation lines a and b do not intersect at all, then since they lie in one acceleration plane they are parallel and the result follows directly from Theorem 55 (b).

We shall therefore consider the case in which an element of

intersection of a and b does exist and we shall denote this element by O .

We shall suppose first that A_x is between a pair of parallel optical lines through A_0 and O in the acceleration plane.

There are however two sub-cases of this according as O is *before* an element of the given optical line through A_0 or *after* an element of it.

It is sufficient however to consider the case where O is *before* an element of this optical line, since the case of its being *after* is quite analogous.

Now by Theorem 58 there exists a definite optical parallelogram in the acceleration plane having the separation line a as one of its diagonal lines and having O as its centre and A_0 as one of its corners.

Let C_0 be the opposite corner to A_0 and let the optical line A_0B_0 intersect the other diagonal line in E_0 while the second optical line through A_0 in the acceleration plane intersects the same diagonal line in G_0 .

Then A_0 , E_0 , C_0 and G_0 are the corners of the optical parallelogram.

Let the separation line b intersect the optical line G_0C_0 in D_0 and let the optical line through D_0 parallel to C_0E_0 intersect A_0E_0 in F_0 , while the optical line through B_0 parallel to C_0E_0 intersects G_0C_0 in H_0 .

Then B_0 , F_0 , D_0 and H_0 are the corners of an optical parallelogram having a pair of side lines in common with the optical parallelogram whose corners are A_0 , E_0 , C_0 and G_0 and having its diagonal line b passing through O the centre of this optical parallelogram, and therefore by Theorem 62 the two optical parallelograms have a common centre O .

Denote the optical parallelogram whose corners are A_0 , E_0 , C_0 and G_0 by P_0 , and the one whose corners are B_0 , F_0 , D_0 and H_0 by Q_0 .

Suppose now that the optical line H_0B_0 intersects the diagonal line A_0C_0 in A_1 and the diagonal line G_0E_0 in G_1 and that the optical line D_0F_0 intersects the diagonal line G_0E_0 in E_1 and the diagonal line A_0C_0 in C_1 .

Then by Theorem 63, A_1 , E_1 , C_1 and G_1 form the corners of an optical parallelogram having also the centre O . Call it P_1 .

Suppose now that the optical line A_1E_1 intersects the diagonal line B_0D_0 in B_1 and the diagonal line H_0F_0 in F_1 , while the optical line G_1C_1 intersects the diagonal line H_0F_0 in H_1 and the diagonal line B_0D_0 in D_1 .

Then as before by Theorem 63, B_1 , F_1 , D_1 and H_1 form the corners of an optical parallelogram Q_1 which bears the same relation to the optical parallelogram P_1 whose corners are A_1 , E_1 , C_1 and G_1 as the optical parallelogram Q_0 to the optical parallelogram P_0 .

This construction may be repeated indefinitely and we obtain a series of parallel optical lines A_0E_0 , A_1E_1 , A_2E_2 , A_3E_3 , etc. intersecting the separation line a in the elements A_0 , A_1 , A_2 , A_3 , etc. and the other diagonal line of the optical parallelogram P_0 in the elements E_0 , E_1 , E_2 , E_3 , etc.

Further, these same optical lines intersect the separation line b in the elements B_0 , B_1 , B_2 , B_3 , etc. and the other diagonal line of the optical parallelogram Q_0 in the elements F_0 , F_1 , F_2 , F_3 , etc.

Again we have another set of parallel optical lines A_1B_0 , A_2B_1 , A_3B_2 , A_4B_3 , etc. and a further set E_1F_0 , E_2F_1 , E_3F_2 , E_4F_3 , etc.

Now by hypothesis O is *before* an element of the optical line A_0E_0 but is *not* an element of it, and since OE_0 is an inertia line we must have E_0 *after* O .

Similarly we must have F_0 *after* O .

But B_0 being *after* A_0 must lie in the α sub-set of A_0 , while A_1 must lie either in the α or β sub-set of B_0 .

A_1 however cannot lie in the α sub-set of B_0 since then it would be *after* A_0 , contrary to the hypothesis that A_0 and A_1 are elements of the same separation line.

Further, A_1 and B_0 cannot coincide, for then B_0 could not be *after* A_0 .

Thus A_1 must lie in the β sub-set of B_0 and must be *before* B_0 .

It follows that A_1E_1 is a before-parallel of A_0E_0 and E_1 must be *before* E_0 since E_1 and E_0 are elements of an inertia line.

Again F_0 must lie either in the α or β sub-set of E_1 and since E_1 cannot be *after* F_0 or coincide with it, it follows that F_0 must be in the α sub-set of E_1 and *after* E_1 .

Thus by Theorem 1 (a) E_1 is *before* F_0 but is not *before* any element outside the sub-set which is *before* F_0 .

But O is *before* F_0 and is outside the sub-set and therefore E_1 is not *before* O .

Further, E_1 cannot coincide with O and therefore must be *after* it since O and E lie in an inertia line.

Thus O is *before* an element of the optical line A_1E_1 and it may similarly be proved that O is *before* an element of each of the parallel optical lines A_2E_2 , A_3E_3 , A_4E_4 , etc.

Further A_2E_2 is a before-parallel of A_1E_1 ,

A_3E_3 " " " " A_2E_2 ,

A_4E_4 " " " " A_3E_3 ,

.....
.....

Now since B_0 and B_1 are elements of a separation line and since B_0 lies in the α sub-set of A_1 , we must have B_1 in the α sub-set of A_1 and *after* A_1 .

Thus

B_1 is *after* A_1 ,
 B_2 „ „ A_2 ,
 B_3 „ „ A_3 ,
.....
.....

If therefore A_x should coincide with any element of the series A_1, A_2, A_3, \dots , an optical line through A_x parallel to A_0B_0 would intersect b in an element which would be *after* A_x .

If however A_x does not coincide with any of these, we may suppose that the optical line through it and parallel to A_0B_0 intersects b in B_x , the inertia line OF_0 in F_x and the inertia line OE_0 in E_x .

Now we have supposed A_x to be *between* a pair of parallel optical lines through A_0 and O in the acceleration plane and so if an optical line be taken through O parallel to A_0E_0 , such optical line must be a before-parallel of A_0E_0 since O is *before* an element of it.

Thus the optical line through A_x parallel to A_0E_0 must be a before-parallel of A_0E_0 and an after-parallel of the optical line through O .

Thus F_x and E_x must be *after* O .

We thus see by Theorem 64 that there can only be a *finite number* of the elements E_1, E_2, E_3, E_4 until we reach one E_n which is *before* E_x .

Let E_n be the one immediately *before* E_x , then E_{n-1} is *after* E_x .

Thus A_xE_x is an after-parallel of A_nE_n and a before-parallel of $A_{n-1}E_{n-1}$ or using B 's instead of E 's we have A_xB_x is an after-parallel of A_nB_n and a before-parallel of $A_{n-1}B_{n-1}$.

Now the optical line A_nB_{n-1} being of the opposite set to A_0B_0 and so also of the opposite set to A_xB_x must intersect A_xB_x in some element, say K .

Then K must be *after* A_n and *before* B_{n-1} .

Also since A_n and A_x are elements of a separation line, K being *after* A_n must also be *after* A_x , while similarly since B_{n-1} and B_x are elements of a separation line and K is *before* B_{n-1} , therefore K must also be *before* B_x .

Thus K is *after* A_x and B_x is *after* K from which it follows that B_x is *after* A_x , which was to be proved.

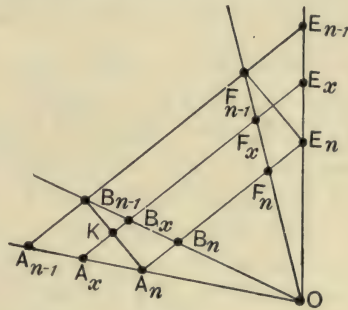


Fig. 14.

Now we started out by considering the case where A_x is between a pair of parallel optical lines through A_0 and O in the acceleration plane; if instead we had taken the case where A_0 is between a pair of parallel optical lines through A_x and O in the acceleration plane, then the supposition that A_x was *after* B_x would, in a similar manner, lead to the conclusion that A_0 was *after* B_0 , contrary to the hypothesis that B_0 is *after* A_0 .

Also since A_x and B_x could not coincide without the separation lines being identical, it follows that we must also in this case have B_x *after* A_x .

Thus the theorem holds in general.

(b) *If A_0 and A_x be two elements in a separation line a which lies in the same acceleration plane with another separation line b which does not intersect a in A_0 , A_x or any element lying between a pair of parallel optical lines through A_0 and A_x in the acceleration plane, and if an optical line through A_0 intersects b in B_0 so that B_0 is before A_0 , then a parallel optical line through A_x will intersect b in an element which is before A_x .*

THEOREM 67.

If two elements A and B lie in one optical line and if two other elements C and D lie in a parallel optical line in the same acceleration plane, then if A be after B and C after D the element of intersection of the general lines AD and BC (which was proved in Theorem 57 to exist) lies between the two given optical lines.

Let a be the optical line containing A and B , and let b be the parallel optical line containing C and D .

Since one of the optical lines must be an after-parallel of the other and since it is immaterial which of them is, we shall suppose that a is an after-parallel of b .

Now the general lines AD and BC cannot *both* be optical lines since two optical lines which intersect a pair of parallel optical lines are themselves parallel and have no element of intersection.

One of them however may be an optical line.

Suppose first that BC is an optical line and that E is the element of intersection of AD and BC .

Then since a is an after-parallel of b and CB is an optical line, therefore B is *after* C .

But C is *after* D and therefore B is *after* D , and since A is *after* B it follows that A is *after* D .

Thus since AD cannot be an optical line and has one element which is *after* another, it must be an inertia line.

Now since C is *after* D and lies in an optical line containing D , it follows that D is in the β sub-set of C ; and since E lies in the second optical line through C in the acceleration plane, it follows by Post. XIV (a) that E must be in the α sub-set of C .

Thus since E cannot be identical with C , it follows that E is *after* C .

Similarly since A is *after* B and A and B lie in an optical line, it follows that A is in the α sub-set of B ; and since E lies in the second optical line through B in the acceleration plane, it follows by Post. XIV (b) that E must be in the β sub-set of B .

Thus since E cannot be identical with B , it follows that E is *before* B .

This proves that E lies between a and b .

Suppose secondly that AD is an optical line and again let E be the element of intersection of AD and BC .

Let the optical line through C parallel to DA intersect a in F .

Then C being *after* D it follows that F must be *after* A and since A is *after* B therefore F must be *after* B .

Now F must be *after* C and therefore lies in the α sub-set of C and is distinct from C , and so by Theorem 1 (a) C is not *before* any element outside the sub-set which is *before* F .

Thus since B is outside the sub-set and *before* F , it follows that C is not *before* B .

But since a is an after-parallel of b , it follows that C is not *after* B .

Thus CB must be a separation line.

Now D cannot be *after* E , for since C is *after* D we should then have C *after* E which is impossible since C and E lie in a separation line.

But since D and E are distinct elements of an optical line, the one must be *after* the other and thus E must be *after* D .

Again E cannot be *after* A , for since A is *after* B we should then have E *after* B which is impossible since E and B are elements of a separation line.

But E must be either *before* or *after* A since E and A are distinct elements of an optical line, and since E cannot be *after*, it must be *before* A .

Thus again in this case E lies between a and b .

Next take the case where one of the two general lines AD and BC is an inertia line and the other a separation line.

If BC is an inertia line, we must have B *after* C since a is an after-parallel of b .

Since then C is *after* D we have B *after* D , and since A is *after* B we must have A *after* D .

This shows that AD could not be a separation line in this case, and so we must instead take BC as a separation line and AD as an inertia line.

If BC be a separation line and E be the element of intersection with AD , then E is neither *before* nor *after* C and also neither *before* nor *after* B .

But E cannot be *before* D , for since D is *before* C we should then have C *after* E which is impossible.

Thus since D and E are distinct elements of an inertia line, we must have E *after* D .

Again E cannot be *after* A , for since A is *after* B we should then have E *after* B which is impossible.

Thus since A and E are distinct elements of an inertia line, we must have E *before* A .

Thus again in this case E lies between a and b .

We shall next take the case where both the general lines AD and BC are inertia lines and E is their element of intersection.

By Theorem 65, if A were *after* D and *before* E then C being *after* D would imply that B was *after* A , contrary to hypothesis; while if D were *after* E and *before* A , then A being *after* B would imply that D was *after* C , contrary again to hypothesis.

Thus since E cannot be identical with either A or D , it follows that E must be *after* D and *before* A and so E lies between a and b .

Finally we have the case where AD and BC are both separation lines and E their element of intersection.

Let c be an optical line through E parallel to a and b .

First suppose, if possible, that c is an after-parallel of a ; then c would also be an after-parallel of b since a is an after-parallel of b .

Thus AD and BC would intersect in an element which was not between a and b and did not lie either in a or b , and so by Theorem 66, A being *after* B would imply that D was *after* C , contrary to hypothesis.

The same would hold if we supposed c to be a before-parallel of b .

Thus c cannot be an after-parallel of a and cannot be identical with a and therefore must be a before-parallel of a .

Also c cannot be a before-parallel of b and cannot be identical with b , and thus c must be an after-parallel of b .

Thus the element E must be *after* an element of b and *before* an element of a and so E lies between a and b .

This exhausts all the possibilities and so we see that the theorem holds in general.

THEOREM 68.

If two elements A and B lie in one optical line and if two other elements C and D lie in a parallel optical line in the same acceleration plane, then if A be after B and if the general lines AD and BC intersect in an element E lying between the parallel optical lines, we must also have C after D .

Let a be the optical line containing A and B , and let b be the parallel optical line containing C and D .

Then one of the optical lines a and b is an after-parallel of the other, but as the demonstration is quite analogous in the two cases we shall only consider that in which a is an after-parallel of b .

We must therefore have E *after* an element of b and *before* an element of a .

Now AD and BC cannot both be optical lines since two optical lines which both intersect a pair of parallel optical lines are themselves parallel and so the element E could not exist.

We may however have one of them an optical line and shall first consider the case in which AD is such.

In this case E is *before* A and A is therefore in the α sub-set of E . It follows by Theorem 1 (a) that E is not *before* any element outside the sub-set which is *before* A .

But B is not in the α sub-set of E but is *before* A and so E is not *before* B .

Also E being *before* A cannot be *after* any element of the optical line a and thus, E being neither *before* nor *after* B , the general line BE must be a separation line.

Thus C can be neither *before* nor *after* E .

But D is *before* E and so if C were *before* D we should have C *before* E , which is impossible.

Further C cannot coincide with D and therefore C must be *after* D .

We shall next consider the case where BC is an optical line.

Then we have B *after* E and since A is *after* B , it follows that A is *after* E and so AE is an inertia line.

Again E is *after* C and so E lies in the α sub-set of C and therefore by Post. XIV (b) D must lie in the β sub-set of C .

Thus since C and D cannot be identical, we must have C *after* D .

We shall next consider the case where one of the general lines BC and AD is an inertia line and the other a separation line.

Now if BC were an inertia line we should have B *after* E and so, since A is *after* B , we should have also A *after* E .

Thus in this case both general lines would be inertia lines and so we must suppose instead that BC is a separation line and AD an inertia line.

Then since E cannot be *before* any element of b and since it must be either *before* or *after* D , it follows that E must be *after* D .

But D cannot be *after* C , for then we should have E *after* C , which is impossible since C and E lie in a separation line.

Thus since C and D cannot be identical, we must have C *after* D .

We have next to consider the cases where the general lines BC and AD are both separation lines and where they are both inertia lines.

The constructions and demonstrations are analogous in both cases up to a certain point.

By Theorem 58 there is an optical parallelogram in the acceleration plane having E as centre and B as one of its corners.

Let C' be the corner opposite to B and let the optical line through C' in the acceleration plane and of the opposite set to AB intersect AB in the element G .

Then GE is the other diagonal line of the optical parallelogram.

Let the second optical line through B in the acceleration plane intersect GE in F .

Then B, F, C' and G are the corners of the optical parallelogram.

Let AE intersect the optical line FC' in D' ; let an optical line through A parallel to BF intersect FC' in H , and let an optical line through D' parallel to $C'G$ intersect BG in I .

Then A, H, D' and I are the corners of an optical parallelogram having a pair of opposite side lines in common with the optical parallelogram whose corners are B, F, C' and G and having one of its diagonal lines AD' passing through E the centre of this optical parallelogram.

It follows from Theorem 62 that these two optical parallelograms have a common centre.

Let AH intersect BC' in A_1 and FG in F_1 and let ID' intersect BC' in C_1 and FG in G_1 .

Then by Theorem 63 the elements A_1, F_1, C_1 and G_1 form the corners of another optical parallelogram with the same centre.

Suppose now first that AE and BE are both separation lines, then EG and EI are both inertia lines, and by hypothesis E is *before* an element of BG and so E must be *before* G and also *before* I .

Also since B and A_1 lie in a separation line and since A is *after* B , it follows that A must also be *after* A_1 .

Thus A_1G_1 must be a before-parallel of BG and so G_1 must be *before* G .

Thus G_1D' must be a before-parallel of GC' , and since C' and D' lie in an optical line we must have C' *after* D' .

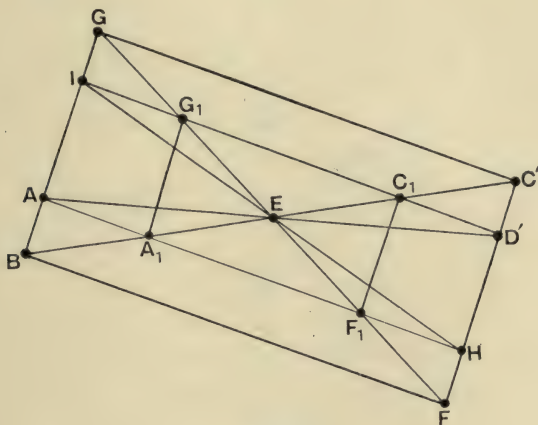


Fig. 15.

Now E being the centre of the optical parallelogram whose corners are B, G, C' and F and being *before* an element of BG must be *after* an element of FC' .

Thus E is between the parallel optical lines BG and FC' .

Now the optical line b containing C and D may either coincide with FC' in which case C is *after* D or else b may be a before-parallel of FC' or an after-parallel of FC' , but in any case b is a before-parallel of a parallel optical line through E .

Thus if D does not coincide with D' we must have either D' between a pair of parallel optical lines through E and D in the acceleration plane or else D between a pair of parallel optical lines through E and D' in the acceleration plane.

Thus by Theorem 66 since C' is *after* D' we must have C *after* D .

Suppose next that AE and BE are both inertia lines, then EG and EI are both separation lines, and by hypothesis E is *before* an element of BG , so E is *before* A and also *before* B .

Also since B and A_1 lie in an inertia line and since B is in the β sub-set of A and distinct from it, therefore A_1 must be in the α sub-set of A , and since B and A are distinct, A and A_1 must also be distinct and therefore A_1 is *after* A .

Thus A_1G_1 must be an after-parallel of AI , and since G_1 and I lie in an optical line we must have G_1 *after* I .

But since G_1 and G lie in a separation line, the one is neither *before* nor *after* the other and so G must also be *after* I .

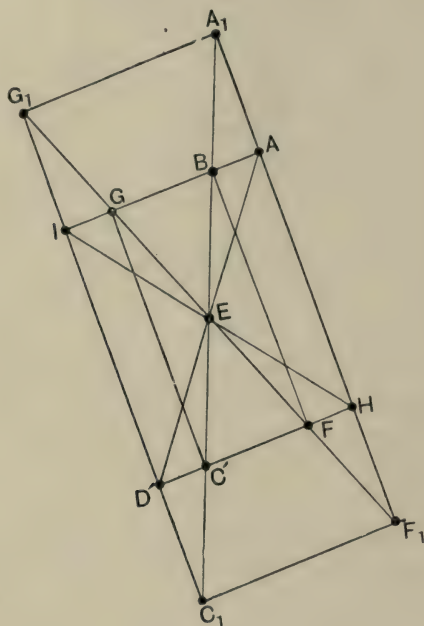


Fig. 16.

Thus GC' must be an after-parallel of ID' , and since C' and D' lie in an optical line we must have C' *after* D' .

From this point the demonstration is similar to that of the case where AE and BE are both separation lines, except that the reference is to Theorem 65 instead of Theorem 66.

This exhausts all the possibilities, and so the theorem holds in general.

THEOREM 69.

If A , B and C be three elements in a separation line and if B be between a pair of parallel optical lines through A and C in an acceleration plane containing the separation line, then B is also between a pair of parallel optical lines through A and C in any other acceleration plane containing the separation line.

Let a be an optical line through A , and c a parallel optical line through C ; both lying in the given acceleration plane, say P , and such that B lies between a and c .

We may suppose that B is *before* an element of a and *after* an element of c without any essential loss of generality.

Let an optical line through B in the acceleration plane, and of the opposite set to a and c , intersect a in D and c in E .

Then D must be *after* B , and E must be *before* B .

Further, since A , B and C lie in a separation line, we must have D *after* A and E *before* C .

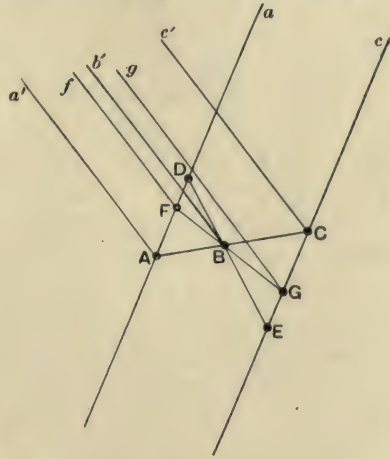


Fig. 17.

Now let Q be any other acceleration plane containing the separation line, and let a' , b' and c' be three parallel optical lines through A , B and C respectively in the acceleration plane Q .

Now the element D is *after* B , an element of the optical line b' , while the optical line a passes through D but does not intersect b' , since then it would have to be identical with the optical line DB which belongs to the opposite set.

Further the optical line a cannot be parallel to b' for since a passes through A it would in that case have to be identical with a' and the acceleration planes P and Q could not be distinct.

Thus each element of a is not *after* an element of b' , and so by Post. XII (b) there is one single element of a , say F , which is neither *after* nor *before* any element of b' .

Thus by Theorem 21 there is one single optical line containing F and such that no element of it is either *before* or *after* any element of b' .

If f be this optical line, then f is a neutral-parallel of b' .

But since a' and b' lie in the acceleration plane Q and are parallel, the one must be an after-parallel of the other and so a' cannot be identical with f .

Thus F must be either *after* or *before* A and cannot be identical with it.

Now the general line FB lies in the acceleration plane P and is clearly a separation line since F is neither *before* nor *after* B .

Let FB intersect the optical line c in G .

Then, by Theorem 45, G is neither *before* nor *after* any element of b' and so if an optical line g be taken through G parallel to b' it will be a neutral-parallel.

Now by Theorem 68, since B lies between the parallel optical lines a and c passing through A and C respectively and lying in the acceleration plane P , it follows that if F be *after* A then C is *after* G ; while if A be *after* F then G is *after* C .

If however F be *after* A , then a' must be a before-parallel of f , and therefore by Theorem 25 (a) a' must be a before-parallel of b' .

Then we shall have also c' an after-parallel of g and therefore by Theorem 25 (b) c' must be an after-parallel of b' .

Thus B will be *after* an element of a' and *before* an element of c' : that is, B will be between the parallel optical lines a' and c' passing through A and C respectively in the acceleration plane Q .

Similarly if F be *before* A , then a' must be an after-parallel of f and therefore, by Theorem 25 (b), a' must be an after-parallel of b' .

We shall in that case have also c' a before-parallel of g and therefore, by Theorem 25 (a), c' must be a before-parallel of b' .

Thus again we shall have B between the parallel optical lines a' and c' passing through A and C respectively in the acceleration plane Q .

Thus the theorem is proved.

REMARKS.

If A , B and C be three elements in an optical or inertia line l , and if B be between a pair of parallel optical lines through A and C in an acceleration plane containing l , then it is easy to see that B is also between a pair of parallel optical lines through A and C in any other acceleration plane containing l .

This follows directly from the consideration that, in this case, of any two of the three elements A , B , C , one is *after* the other.

We accordingly introduce the following definition.

Definition. If three distinct elements lie in a general line and if one of them lies between a pair of parallel optical lines through the other two in an acceleration plane containing the general line, then the element which is between the parallel optical lines will be said to be *linearly between* the other two elements.

The above definition is so framed as to apply to all three types of general line and for this reason is more complicated than it need be if we were dealing only with optical or inertia lines.

For the case of elements lying in either of these types of general line, one element is linearly between two other elements if it be *after* the one and *before* the other.

In the case of elements lying in a separation line, however, no one is either *before* or *after* another and so we have to fall back on our definition involving parallel optical lines.

The distinction between the three cases is interesting.

Thus if the three elements A , B and C lie in a general line a , and if B be linearly between A and C , then, in case a be an inertia line, we must have either B *after* A and C *after* B or else B *after* C and A *after* B , and similarly when a is an optical line.

If a be an inertia line and B be *after* A and C *after* B , then B will be *before* elements of both optical lines through C and *after* elements of both optical lines through A in any acceleration plane containing a .

If a be an optical line and B be *after* A and C *after* B , then, *apart from a itself*, there is only one optical line through any element of a in any acceleration plane containing a , and so we should have B *before* an element of the optical line through C and *after* an element of the parallel optical line through A .

If a be a separation line however, we should have B *before* an element of one of the optical lines through C and *after* an element of the parallel optical line through A and also *after* an element of the second optical line through C and *before* an element of the parallel optical line through A .

The distinctions are perhaps exhibited more clearly by the following figures:

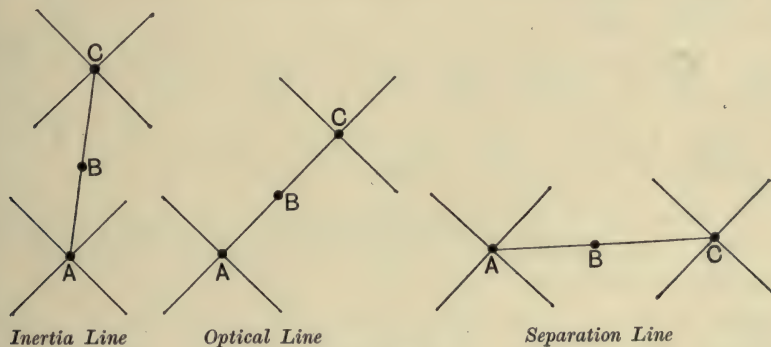


Fig. 18.

From Theorem 69 it follows that the property of one element being linearly between two others is independent of the particular acceleration

plane in which the elements are considered as lying and so may be regarded as a relation of the one element to the other two.

This relation has been defined in terms of the relations *before* and *after*, not only for the cases where the three elements considered are such that of any two of them one is *after* the other; but also for the case of elements in a separation line when this is no longer so.

It is thus possible to state certain general results which hold for all three types of general line involving the conception linearly between.

Peano has given some eleven axioms of the *straight line* which are as follows:

- (1) There is at least one point.
- (2) If A is any point, there is a point distinct from A .
- (3) If A is a point, there is no point lying between A and A .
- (4) If A and B are distinct points, there is at least one point lying between A and B .
- (5) If the point C lies between A and B , it also lies between B and A .
- (6) The point A does not lie between the points A and B .

Definition. If A and B are points, the symbol AB represents the class of points such as C with the property that C lies between A and B .

Definition. If A and B are points, the symbol $A'B$ represents the class of points such as C with the property that B lies between A and C . Thus $A'B$ is the prolongation of the line beyond B , and $B'A$ its prolongation beyond A .

- (7) If A and B are distinct points, there exists at least one member of $A'B$.
- (8) If A and D are distinct points and C is a member of AD and B of AC , then B is a member of AD .
- (9) If A and D are distinct points and B and C are members of AD , then either B is a member of AC , or B is identical with C , or B is a member of CD .
- (10) If A and B are distinct points and C and D are members of $A'B$, then either C is identical with D , or C is a member of BD , or D is a member of BC .
- (11) If A, B, C, D are points and B is a member of AC and C of BD , then C is a member of AD .

Definition. The straight line possessing A and B , symbolized by str. (A, B) , is composed of the three classes $A'B$, AB , $B'A$ together with the points A and B themselves.

Of these axioms the writer has succeeded in proving nos. (6) and (9) from the others, so that they are really redundant*.

It is easy to see, with our definition of linearly between, that corresponding results hold for all three types of "general line."

As regards axioms (1) and (2) which we shall express thus:

(1) *There is at least one element,*

and (2) *If A be any element there is an element distinct from A ,* the first follows from our preliminary statement on page 10, while the second follows directly from Posts. II and I and also from Post. V.

As regards axiom (3) we shall put it in the form:

(3) *If A is an element, there is no element lying linearly between A and A .*

This follows from the definition of linearly between.

(4) *If A and B are distinct elements, there is at least one element lying linearly between A and B .*

From our remarks at the end of Theorem 35 it appears that there are an infinite number of acceleration planes containing any two distinct elements and accordingly any two distinct elements lie in a general line.

If A and B lie in an optical line, then Theorem 11 shows that there is at least one element which is *after* the one and *before* the other and is therefore linearly between them.

If A and B lie in an inertia line, the same result follows from Theorem 39; while if they lie in a separation line, it follows from Theorem 41.

(5) *If the element C lies linearly between A and B , it also lies linearly between B and A .*

This follows from the definition of linearly between.

(6) *The element A does not lie linearly between the elements A and B .*

This follows from the definition of what is meant by an element lying between a pair of parallel optical lines in an acceleration plane. According to this definition the element must not lie in either optical line.

* *Messenger of Mathematics*, vol. XIII, pp. 121-123 and 134.

(7) *If A and B are distinct elements, there is at least one element such that B lies linearly between it and A.*

If A and B lie in an optical line or an inertia line, one of them must be *after* the other.

If it be the element A which is *after* B, then Theorems 7 and 38 show that there is at least one element of the general line which is *before* B, and so B lies linearly between it and A.

Similarly if A be *before* B there is an element of the general line which is *after* B, and so B is linearly between it and A.

If A and B lie in a separation line, the result follows from Theorem 43.

(8) *If A and D are distinct elements and C is linearly between A and D, and B linearly between A and C, then B is linearly between A and D.*

This is readily seen to be true if we take a set of parallel optical lines *a, b, c* and *d* through A, B, C and D respectively in any acceleration plane containing the four elements.

Let these optical lines intersect an optical line *f* of the opposite set in A', B', C' and D' respectively.

Remembering that Post. III must be satisfied, it is clear that we must have either:

(i) *C' after D' and A' after C' together with B' after C' and A' after B' ;*

or (ii) *C' before D' and A' before C' together with B' before C' and A' before B'.*

In the first case it follows by Post. III that B' is *after* D' and consequently since B' is *before* A' we have B linearly between A and D.

Similarly in the second case we have B' *before* D' and *after* A', and therefore again B linearly between A and D.

(9) *If A and D are distinct elements and B and C are each linearly between A and D, then either B is linearly between A and C or B is identical with C or B is linearly between C and D.*

This result may be deduced in a similar manner to the last.

We must have either

(i) *B' after D' and A' after B' together with C' after D' and A' after C' ;*

or (ii) *B' before D' and A' before B' together with C' before D' and A' before C'.*

Then the elements B' and C' must either be identical or else the one is *after* the other.

In the first case if B' be *after* C' , since also B' is *before* A' , we have B linearly between A and C .

If B' is identical with C' , then B is identical with C .

If C' be *after* B' , then since also D' is *before* B' we have B linearly between C and D .

Similarly in the second case we must either have B linearly between A and C or B identical with C or B linearly between C and D .

(10) *If A and B are distinct elements and if B is linearly between A and C and also linearly between A and D , then either C is identical with D , or C is linearly between B and D , or D is linearly between B and C .*

This result may also be deduced in a similar way. We must have either:

- (i) B' *after* C' and A' *after* B' together with B' *after* D' ;
- or (ii) B' *before* C' and A' *before* B' together with B' *before* D' .

Then the elements C' and D' must either be identical or else the one is *after* the other.

In the first case if C' is *after* D' , then since C' is *before* B' we have C linearly between B and D .

If C' is identical with D' , then C is identical with D .

If D' is *after* C' , then since D' is *before* B' we have D linearly between B and C .

Similarly in the second case we must either have C linearly between B and D or C identical with D , or D linearly between B and C .

(11) *If A, B, C, D are elements and B is linearly between A and C , and C is linearly between B and D , then C is linearly between A and D .*

This result may also be deduced in a similar way. We must have either:

- (i) B' *after* C' and A' *after* B' together with C' *after* D' ;
- or (ii) B' *before* C' and A' *before* B' together with C' *before* D' .

In the first case, since B' is *after* C' and A' *after* B' , it follows by Post. III that A' is *after* C' , and so C must be linearly between A and D .

Similarly in the second case we must also have C linearly between A and D .

Thus all these axioms of Peano hold for the general line.

THEOREM 70.

(a) *If A_0 and A_x be two elements in a general line a which lies in the same acceleration plane with another general line b which intersects a in the element C such that either A_0 is linearly between C and A_x , or A_x is linearly between C and A_0 , and if an optical line through A_0 intersects b in B_0 so that B_0 is after A_0 , then a parallel optical line through A_x will intersect b in an element which is after A_x .*

We have already proved special cases of this in Theorems 65 and 66, and have now to prove the general theorem.

The optical line through A_x parallel to A_0B_0 must intersect b since b intersects A_0B_0 in B_0 .

Let the element of intersection of this optical line through A_x with b be B_x .

Then B_x cannot be identical with A_x , for then the general lines a and b would have two distinct elements C and A_x in common and would therefore be identical, which is impossible since a and b intersect by hypothesis.

Further, if A_x were *after* B_x the general lines a and b would intersect in some element between the parallel optical lines (Theorem 67).

That is to say in some element linearly between A_0 and A_x .

But a and b have only one element C in common, so that if A_x were *after* B_x we should require C to be linearly between A_0 and A_x , contrary to the hypothesis that either A_0 is linearly between C and A_x or A_x is linearly between C and A_0 .

Thus B_x must be *after* A_x .

(b) *If A_0 and A_x be two elements in a general line a which lies in the same acceleration plane with another general line b which intersects a in the element C such that either A_0 is linearly between C and A_x , or A_x is linearly between C and A_0 , and if an optical line through A_0 intersects b in B_0 so that B_0 is before A_0 , then a parallel optical line through A_x will intersect b in an element which is before A_x .*

The following five theorems (71 to 75 inclusive) are special cases of Theorems 76 and 77, but as the proofs of the general theorems are reduced to depend on these special cases the latter are treated separately.

THEOREM 71.

If A , B and C be three distinct elements in an acceleration plane, of which A and B lie in an optical line, but C does not lie in it, and if D be an element linearly between A and B while E is an element linearly

between B and C , then there exists an element which lies both linearly between C and D and also linearly between A and E .

It will be sufficient to consider the case where A is *after* B , since the case where B is *after* A is quite analogous.

Since E is linearly between B and C , therefore E is between the optical line AB and a parallel optical line through C .

Thus since A is *after* B this optical line through C intersects the general line AE in some element, say G , such that C is *after* G (Theorem 68).

But since D is linearly between A and B and A is *after* B , therefore A is *after* D and D is *after* B .

But A being *after* D and C *after* G it follows by Theorem 67 that the general lines AG and DC intersect in some element, say F , which is between the optical lines AD and CG .

That is, F lies linearly between C and D .

But now D is *after* B and E is linearly between C and B , and so by Theorem 70 an optical line through E parallel to BD will intersect CD in some element, say H , such that H is *after* E .

But since A is *after* D and H is *after* E , therefore by Theorem 67 the element F of intersection of AE and DH is between the parallel optical lines AD and HE .

That is, F lies linearly between A and E , and we have already shown that F is linearly between C and D .

This proves the theorem.

THEOREM 72.

If A , B and C be three distinct elements in an acceleration plane, of which A and B lie in an optical line, but C does not lie in it, and if E be an element linearly between B and C while I is an element linearly between C and A , then there exists an element which lies both linearly between A and E and also linearly between B and I .

It will again be sufficient to consider only the case where A is *after* B .

As in Theorem 71, since A is *after* B and E is linearly between B and C , therefore the general line AE intersects an optical line through C parallel to AB in some element, say G , such that C is *after* G .

Again, since A is *after* B and I is linearly between C and A , it follows in a similar manner that the general line BI intersects the optical line CG in some element, say J , such that J is *after* C .

Thus since J is *after* C and C is *after* G , we have J *after* G .

But since A is *after* B and J *after* G , therefore by Theorem 67 the general lines AG and BJ intersect in some element, say F , which is between the parallel optical lines AB and JG .

But since also C is *after* G , the general line CF must intersect AB in some element, say D , such that A is *after* D .

Also since J is *after* C and F between the optical lines JC and AB , therefore D is *after* B .

Thus D is linearly between A and B , and so by Theorem 71, F is linearly between A and E and in a similar manner F is linearly between B and I .

This proves the theorem.

THEOREM 73.

If A , B and C be three distinct elements in an acceleration plane, of which A and B lie in an optical line, but C does not lie in it, and if D be an element linearly between A and B , while F is an element linearly between C and D , then there exists an element, say E , which lies linearly between B and C and such that F lies linearly between A and E .

As in the previous two theorems it will be sufficient to consider only the case where A is *after* B .

Then A will be *after* D and D *after* B .

Now since F is linearly between C and D , therefore F is between the optical line AD and a parallel optical line through C and so since A is *after* D it follows by Theorem 68 that this optical line through C intersects the general line AF in some element, say G , such that C is *after* G .

But since C is *after* G and A is *after* B , therefore the general lines AG and BC intersect in some element, say E , such that E is between AB and CG .

That is, E is linearly between B and C .

But since D is *after* B , it follows by Theorem 70 that an optical line through E parallel to BD will intersect CD in some element, say H , such that H is *after* E .

But since A is *after* D , it follows that the general lines AE and HD intersect in an element which is between AD and HE .

Thus F must be between AD and HE ; that is, F must be linearly between A and E .

We have however already shown that E is linearly between B and C , so that the theorem is proved.

THEOREM 74.

If A , B and C be three distinct elements in an acceleration plane, of which A and B lie in an optical line, but C does not lie in it, and if E be an element linearly between B and C , while F is an element linearly between A and E , then there exists an element, say D , which lies linearly between A and B and such that F lies linearly between C and D .

It will again be sufficient to consider only the case where A is *after* B .

As in Theorem 71, since A is *after* B and E is linearly between B and C , therefore the general line AE intersects an optical line through C parallel to AB in some element, say G , such that C is *after* G .

Also E is linearly between A and G .

But F is linearly between A and E and so, by the analogue of Peano's eighth axiom, F is linearly between A and G .

Thus F is between the parallel optical lines AB and CG , and since C is *after* G it follows that CF intersects AB in an element, say D , such that A is *after* D .

Further F is linearly between C and D .

Now F being linearly between A and E is between AD and a parallel optical line through E , and since A is *after* D the optical line through E parallel to AD will intersect DF in some element, say H , such that H is *after* E .

But since E is linearly between B and C , therefore, by Theorem 70, D must be *after* B .

Thus D is linearly between A and B while F is linearly between C and D .

Thus the theorem is proved.

THEOREM 75.

If A , B and C be three distinct elements in an acceleration plane, of which A and B lie in an optical line but C does not lie in it, and if E be an element linearly between B and C , while F is an element linearly between A and E , then there exists an element, say I , which lies linearly between C and A and such that F lies linearly between B and I .

It will again be sufficient to consider the case where A is *after* B .

Since F is linearly between A and E , therefore F is between AB and a parallel optical line through E , and since A is *after* B this optical line through E will intersect the general line BF in some element K , such that K is *after* E .

But since E is linearly between B and C , therefore an optical line

through C parallel to EK will intersect BK in some element J , such that J is *after* C .

But since J is *after* C and A *after* B , therefore the general lines AC and BJ will intersect in some element, say I , such that I is between AB and JC .

Thus I is linearly between C and A and we have also E linearly between B and C , and so by Theorem 72, F must be linearly between B and I .

Thus the theorem is proved.

THEOREM 76.

If A , B and C be three elements in an acceleration plane which do not all lie in one general line and if D be an element linearly between A and B , while E is an element linearly between B and C , there exists an element which lies both linearly between A and E and linearly between C and D .

By combining Theorems 71 and 72 we see that the above holds for the special case where one of the general lines AB , BC or CA is an optical line.

In considering the remaining cases it is evident that an optical line through A in the acceleration plane will intersect the general line BC in some element M which cannot coincide with either B or C .

It will be sufficient to consider the cases where A is *after* M , as the cases where A is *before* M are quite analogous.

Three special cases have to be considered. We may have either

- (i) M linearly between B and C ,
- or (ii) B linearly between M and C ,
- or (iii) C linearly between M and B .

We shall take these in order.

CASE (i). Since the general lines AB , BC and CA are none of them optical lines, they must be inertia or separation lines and so it is evident that if we take optical lines through B , C and D parallel to AM then the general line BA must intersect the optical line through C in some element, say G , while the general line CA must intersect the optical line through B in some element H and the optical line through D in some element, say J .

Then since M is linearly between B and C and since A is *after* M , therefore, by Theorem 70, G is *after* C .

Similarly H is *after* B , and so by Theorem 67 the element A is between CG and BH .

That is, A is linearly between B and G .

But D is linearly between B and A , and so by the analogue of Peano's axiom (8) we have D linearly between B and G .

That is, D is between the parallel optical lines CG and BH , and so by Theorem 68, since G is *after* C and CD is not parallel to HB , we have CD intersecting BH in some element I such that I is *after* B .

But since I is *after* B and M is linearly between C and B , therefore by Theorem 70 the optical line AM must intersect CI in some element, say F_1 , such that F_1 is *after* M .

Similarly, since H is *after* B and D is linearly between A and B we must have J *after* D .

But since G is *after* C and J *after* D , therefore by Theorem 67 we must have A between CG and DJ .

That is, A linearly between G and D .

Thus the optical line F_1A must be either an after-parallel of CG and a before-parallel of DJ , or else F_1A must be a before-parallel of CG and an after-parallel of DJ .

In either case F_1 is linearly between C and D , and since G is *after* C we must have A *after* F_1 .

Thus F_1 is linearly between A and M and also linearly between C and D .

Consider now the element E .

Since E is linearly between B and C and M is also linearly between B and C , therefore either

(1) E coincides with M ,

or (2) E is linearly between C and M ,

or (3) E is linearly between M and B .

(1) If E coincides with M , then the element F_1 is linearly between A and E and also linearly between C and D .

(2) If E is linearly between C and M , then since AM is an optical line and F_1 is linearly between A and M , therefore by Theorem 71 there is an element, say F , which is linearly between A and E and also linearly between C and F_1 .

But since F_1 is linearly between C and D , therefore F is linearly between C and D .

(3) If E is linearly between M and B , then since AM is an optical line and D is linearly between A and B , therefore by Theorem 72 there

is an element, say K , which is linearly between A and E and also linearly between M and D .

But again, by Theorem 71, since F_1 is linearly between A and M , and K linearly between M and D , there exists an element, say F , which is both linearly between F_1 and D and also linearly between A and K .

Since, however, F_1 is linearly between C and D and F is linearly between F_1 and D , therefore F is linearly between C and D .

Further, since K is linearly between A and E , while F is linearly between A and K , therefore F is linearly between A and E .

CASE (ii). Here we have B linearly between M and C , and so since D is linearly between A and B it follows by Theorem 74 that there exists an element, say F_1 , which lies linearly between A and M and such that D is linearly between C and F_1 .

But since B is linearly between M and C while E is linearly between B and C , therefore it follows that E is linearly between C and M .

Thus since F_1 is linearly between A and M and AM is an optical line, it follows by Theorem 71 that there exists an element, say F , which lies both linearly between C and F_1 and also linearly between A and E .

But since B is linearly between M and C while E is also linearly between M and C , therefore either B is linearly between M and E , or B is identical with E , or B is linearly between E and C .

But since E is linearly between B and C therefore E cannot be identical with B nor can B be linearly between E and C .

Thus we must have B linearly between M and E .

But since F_1 is linearly between A and M and since AM is an optical line, therefore by Theorem 71 there exists an element, say K , which lies both linearly between E and F_1 and also linearly between A and B .

Thus since F is linearly between A and E , and K linearly between E and F_1 and since AF_1 is an optical line, therefore by Theorem 72 there exists an element which is linearly between A and K and also linearly between F_1 and F .

But this element is the element of intersection of AB and CF_1 and must therefore be the element D .

Thus D is linearly between F_1 and F .

But F is linearly between C and F_1 and D is also linearly between C and F_1 and so either F is linearly between C and D , or F is identical with D , or F is linearly between D and F_1 .

Since however D is linearly between F_1 and F we cannot have either F identical with D nor can we have F linearly between D and F_1 .

Thus F must be linearly between C and D and we have already proved that F is also linearly between A and E .

CASE (iii). Here we have C linearly between M and B and since A is *after* M it follows by Theorem 70 that an optical line through C parallel to MA will intersect AB in some element, say M' , such that M' is *after* C .

Also since CM' must be either a before-parallel of MA and an after-parallel of a parallel optical line through B , or else CM' is an after-parallel of MA and a before-parallel of a parallel optical line through B , it follows that M' is linearly between B and A .

Now we have seen that the cases where A is *before* M are quite analogous to those where A is *after* M .

If now in order to compare this with case (i) we change our notation and put A' for C , B' for B , C' for A , E' for D and D' for E , we have D' linearly between A' and B' and also E' linearly between B' and C' and the optical line $A'M'$ intersecting $B'C'$ in M' so that M' is linearly between B' and C' , while A' is *before* M' .

Thus by case (i) there exists an element F which is linearly between A' and E' and also linearly between C' and D' .

That is, there is an element F which is both linearly between C and D and also linearly between A and E .

Thus the theorem holds in general.

THEOREM 77.

If A , B and C be three elements in an acceleration plane which do not all lie in one general line, and if D be an element linearly between A and B while F is an element linearly between C and D , there exists an element, say E , which is linearly between B and C and such that F is linearly between A and E .

By combining Theorems 73, 74 and 75 we see that the above holds for the special case where one of the general lines AB , BC or CA is an optical line.

As regards the remaining cases, it is evident that an optical line through A in the acceleration plane will intersect the general line BC in some element M which cannot coincide either with B or C .

It will be sufficient to consider the cases where A is *after* M , as the cases where A is *before* M are quite analogous.

Three special cases have to be considered.

We may have either

- (i) M linearly between B and C ,
- or (ii) B linearly between M and C ,
- or (iii) C linearly between M and B .

We shall take these in order.

CASE (i). Since A is *after* M and M is linearly between B and C , therefore an optical line through C parallel to MA will intersect the general line BA in some element G such that G is *after* C (Theorem 70).

Also MA must be either an after-parallel of CG and a before-parallel of a parallel optical line through B , or else MA is a before-parallel of CG and an after-parallel of a parallel optical line through B .

In any case A must be linearly between B and G .

But D is linearly between A and B , and so we have D linearly between B and G .

Also since A is linearly between B and G while D is also linearly between B and G , therefore we must have either A linearly between D and G , or A identical with D , or A linearly between D and B .

But since D is linearly between A and B , we cannot have either A identical with D or A linearly between D and B .

Thus A is linearly between G and D .

Now the general line CD must intersect the optical line through B parallel to MA in some element I since it cannot be parallel to it, and since G is *after* C and D is linearly between G and B it follows by Theorem 68 that I must be *after* B .

Then since D is linearly between A and B and therefore between the parallel optical lines MA and BI , and since I is *after* B , it follows that the element of intersection, say F_1 , of AM and CI must be such that A is *after* F_1 .

Again, since I is *after* B and M is linearly between B and C therefore by Theorem 70 F_1 must be *after* M .

Thus F_1 is linearly between A and M .

But since A is linearly between G and D we must have AF_1 either an after-parallel of CG and a before-parallel of a parallel optical line through D , or else AF_1 is a before-parallel of CG and an after-parallel of a parallel optical line through D .

In either case F_1 is linearly between C and D .

Consider now the element F .

Since F is linearly between C and D , and F_1 is also linearly between C and D , therefore either

- (1) F coincides with F_1 ,
 or (2) F is linearly between C and F_1 ,
 or (3) F is linearly between F_1 and D .

(1) If F coincides with F_1 then the element M is identical with the element E which is linearly between B and C while F is linearly between A and E .

(2) If F is linearly between C and F_1 , then since F_1 is linearly between A and M , and since AM is an optical line, it follows by Theorem 73 that there exists an element, say E , which lies linearly between C and M and such that F is linearly between A and E .

But since E is linearly between C and M , while M is linearly between B and C , therefore E is linearly between B and C .

(3) If F is linearly between F_1 and D , then since F_1 is linearly between A and M and since AM is an optical line it follows by Theorem 73 that there exists an element, say K , such that K is linearly between M and D while F is linearly between A and K .

But since D is linearly between A and B , while K is linearly between M and D , and since AM is an optical line, it follows by Theorem 75 that there exists an element, say E , such that E is linearly between M and B while K is linearly between A and E .

But since F is linearly between A and K while K is linearly between A and E therefore F is linearly between A and E .

Also since E is linearly between M and B while M is linearly between B and C therefore E is linearly between B and C .

CASE (ii). Here we have B linearly between M and C and since also D is linearly between A and B and since AM is an optical line, it follows by Theorem 74 that there exists an element, say F_1 , such that F_1 is linearly between A and M while D is linearly between C and F_1 .

Consider now the element F .

Since F is linearly between C and D while D is linearly between C and F_1 , therefore F is linearly between C and F_1 .

But since also F_1 is linearly between A and M and since AM is an optical line, it follows by Theorem 73 that there exists an element, say E , which is linearly between C and M and such that F is linearly between A and E .

Now since F is linearly between C and F_1 while D is also linearly between C and F_1 , therefore either D is linearly between F and F_1 , or D coincides with F , or D is linearly between C and F .

But since F is linearly between C and D , we cannot have either D linearly between C and F or D coincident with F .

Thus D is linearly between F and F_1 .

Since further F is linearly between A and E and since AF_1 is an optical line, it follows by Theorem 75 that there exists an element, say K , which is linearly between E and F_1 and such that D is linearly between A and K .

Also since K is linearly between E and F_1 , while F_1 is linearly between A and M , and since AM is an optical line, it follows by Theorem 73 that there is an element, say B' , which is linearly between E and M and such that K is linearly between A and B' .

But the general line AK is identical with the general line AD and so B' must be identical with B .

Thus B is linearly between E and M .

But E is linearly between M and C while B also is linearly between M and C and so either E is linearly between C and B , or E is identical with B , or E is linearly between B and M .

Since however B is linearly between E and M we cannot have either E identical with B or E linearly between B and M .

Thus E must be linearly between B and C and we have already shown that F is linearly between A and E .

CASE (iii). Here we have C linearly between M and B and so, by Theorem 70, since A is *after* M an optical line through C parallel to MA will intersect AB in some element, say M' , such that M' is *after* C .

Further, the optical line CM' must be either a before-parallel of MA and an after-parallel of a parallel optical line through B , or else CM' is an after-parallel of MA and a before-parallel of a parallel optical line through B .

In either case M' must be linearly between A and B .

But D is also linearly between A and B and so we must have either D linearly between A and M' , or D identical with M' , or D linearly between M' and B .

First suppose D linearly between A and M' .

Then since F is linearly between C and D and since CM' is an optical line, therefore by Theorem 74 there exists an element, say F_1 , which is linearly between C and M' and such that F is linearly between A and F_1 .

Secondly, suppose D to be identical with M' .

Then the general line CD is identical with the optical line CM' and

F may then be taken as identical with F_1 and so F_1 is linearly between C and M' .

Thirdly, suppose D linearly between M' and B .

Then since M' is linearly between A and B it follows that M' is linearly between A and D .

Also since F is linearly between C and D it follows by Theorem 76 that there exists an element, say F_1 , which is both linearly between A and F and also linearly between C and M' .

Thus in all three cases we have an element F_1 linearly between C and M' and lying in the general line AF .

But since M' is *after* C , therefore F_1 is *after* C and *before* M' .

Since however M' is linearly between A and B and F_1 is *before* M' , therefore by Theorem 70 an optical line through B parallel to $M'C$ will intersect AF_1 in some element, say G , such that G is *before* B .

But since F_1 is *after* C and B *after* G , therefore by Theorem 67 the general lines BC and GF_1 intersect in some element, say E , which is between the parallel optical lines CM' and GB .

Thus E is linearly between B and C .

But since D is linearly between A and B , therefore by Theorem 76 there is an element which is both linearly between A and E and also linearly between C and D .

But since the general lines AE and CD have only the element F in common, it follows that F is linearly between A and E .

Thus the theorem holds in all cases.

REMARKS.

Peano has given the following three axioms of the plane:

(12) If r is a straight line, there exists a point which does not lie on r .

(13) If A, B, C are three non-collinear points and D lies on the segment BC , and E on the segment AD , there exists a point F on both the segment AC and the prolongation $B'E$.

(14) If A, B, C are three non-collinear points and D lies on the segment BC , and F on the segment AC , there exists a point E lying on both the segments AD and BF .

Now since there is always an element outside any general line it follows that the analogue of Peano's axiom (12) holds for our geometry.

Further, provided the three elements A, B, C lie in an acceleration plane, Theorem 76 corresponds to Peano's axiom (14) while Theorem 77 corresponds to his axiom (13).

Also Theorem 47 corresponds to the axiom of parallels in Euclidean geometry so far as an acceleration plane is concerned.

An acceleration plane however differs from a Euclidean plane, since there are three types of general line in the former and only one type of straight line in the latter.

Further, although closed figures exist in an acceleration plane, there is no closed figure which corresponds to a circle.

How this comes about will be seen hereafter.

THEOREM 78.

If A , B and C be three elements in an acceleration plane which do not all lie in one general line and if D be an element linearly between A and B while DE is a general line through D parallel to BC and intersecting AC in the element E , then E is linearly between A and C .

In the first place E cannot be identical with A for then the general line DE would be identical with the general line BA and would therefore intersect BC .

Again E cannot be identical with C for once more BC and DE would intersect.

Thus we must either have C linearly between A and E , or A linearly between C and E , or E linearly between A and C .

If C were linearly between A and E then since D is linearly between A and B it would follow by Theorem 76 that there existed an element which was both linearly between B and C and linearly between E and D .

Thus in this case also BC and DE would intersect.

Next if A were linearly between C and E , then since D is linearly between A and B it would follow similarly by Theorem 77 that BC and DE must intersect.

Thus the only possibility is that E is linearly between A and C .

THEOREM 79.

If three parallel general lines a , b and c in one acceleration plane intersect a general line d_1 in A_1 , B_1 and C_1 respectively and intersect a second general line d_2 in A_2 , B_2 and C_2 respectively, then if B_1 is linearly between A_1 and C_1 we shall also have B_2 linearly between A_2 and C_2 .

If A_1 should be identical with A_2 the result follows directly from Theorem 78.

Similarly it follows directly if C_1 should be identical with C_2 .

If B_1 should be identical with B_2 the following method is still valid.

The general line C_1A_2 cannot be identical with the general line c and therefore C_1A_2 must intersect the general line b (which is parallel to c) in some element, say B' .

Then since B_1 is linearly between A_1 and C_1 , it follows by Theorem 78 that B' is linearly between C_1 and A_2 .

Similarly, since B' is linearly between A_2 and C_1 , it follows that B_2 is linearly between A_2 and C_2 .

Definition. If two parallel general lines in an acceleration plane be both intersected by another pair of parallel general lines then the four general lines will be said to form a *general parallelogram in the acceleration plane*.

It will be seen hereafter that it is necessary to extend the meaning of the phrase *general parallelogram* to the case of figures which do not lie in an acceleration plane and so the words "*in an acceleration plane*" are important.

The general lines which form a general parallelogram in an acceleration plane will be called the *side lines* of the general parallelogram.

A pair of parallel side lines will be said to be *opposite*.

The elements of intersection of pairs of side lines which are not parallel will be called the *corners* of the general parallelogram.

A pair of corners which do not lie in the same side line will be said to be *opposite*.

A general line passing through a pair of opposite corners will be called a *diagonal line* of the general parallelogram.

It is clear that a general parallelogram in an acceleration plane has two diagonal lines.

Further it is clear that an optical parallelogram is a particular case of a general parallelogram in an acceleration plane.

THEOREM 80.

If we have a general parallelogram in an acceleration plane, then :

(1) *The two diagonal lines intersect in an element which is the mean of either pair of opposite corners.*

(2) *A general line through the element of intersection of the diagonal lines and parallel to either pair of side lines, intersects either of the other side lines in an element which is the mean of the pair of corners through which that side line passes.*

We shall first consider the cases in which one pair of opposite side lines are optical lines.

Let A, B, C, D be the corners of the general parallelogram and let AC and BD be the one pair of opposite side lines which are optical, while AB and CD are the other pair of opposite side lines.

We shall suppose that AB and CD are not optical lines, since, in case they are, the result follows directly from the definitions of the mean of a pair of elements.

Let the second optical line through B in the acceleration plane intersect the optical line AC in G , and let the second optical line through D in the acceleration plane intersect AC in F .

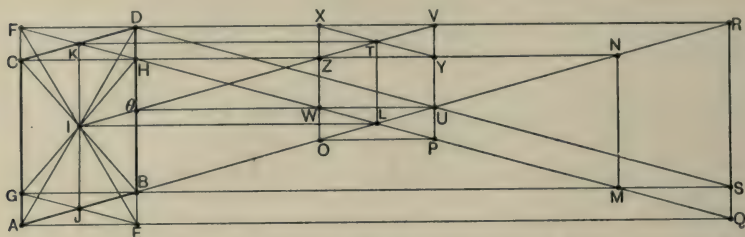


Fig. 19.

Further let AE be an optical line through A parallel to GB and intersecting BD in E , and let CH be an optical line through C parallel to FD and intersecting BD in H .

Then A, E, D and F form the corners of an optical parallelogram as do also F, D, C and H and also G, B, A and E .

Further, if C and G do not coincide then C, H, G and B also form the corners of an optical parallelogram.

Let the diagonal lines AD and EF intersect in I , the diagonal lines AB and EG in J and the diagonal lines CD and HF in K .

Then by Theorem 61, J, I and K lie in an optical line parallel to AC and BD , and if C and G are distinct the centre of the optical parallelogram whose corners are C, G, B and H also lies in the same optical line.

Now since AB and CD are parallel it follows that AB and FH must intersect in some element, say L .

Also, since AE is an optical line while FH is not, it follows that AE and FH intersect in some element, say Q .

Similarly GB and FH intersect in some element, say M .

Again, since FD is an optical line and AB is not, it follows that FD and AB intersect in some element, say R , and similarly CH and AB intersect in some element, say N .

If now the second optical line through Q in the acceleration plane

intersects FR in R' , then F , A , Q and R' are the corners of an optical parallelogram.

But if we consider the optical parallelogram whose corners are F , C , H , D , this has the diagonal line FH which is identical with the diagonal line FQ and so the diagonal lines of the other kind, namely CD and AR' , do not intersect.

Now by hypothesis CD and AB are parallel, and so AB and AR' must be the same general line.

It follows that R' must be identical with R .

Thus A , Q , R , F are the corners of an optical parallelogram whose centre is L .

If we consider the case where C and G do not coincide we may prove in a similar manner that H , B , M , N are also the corners of an optical parallelogram having L as centre.

Confining our attention for the present to the case where C and G are distinct, let CB and GH intersect in I' .

Then we have seen that I' lies in the optical line JK as does also I .

Further, since the optical parallelograms whose corners are A , E , D , F and A , Q , R , F have a pair of opposite side lines in common, therefore their centres I and L lie in an optical line parallel to FR .

Also since the optical parallelograms whose corners are G , B , H , C and B , M , N , H have a pair of opposite side lines in common, therefore their centres I' and L lie in an optical line parallel to HN .

That is, parallel to FR .

Thus I and I' both lie in the optical line through L parallel to FR and as we have seen they also both lie in the optical line JK .

Thus I' must be identical with I .

But since CB and GH intersect in I , therefore I is the mean of B and C .

Also since AD and EF intersect also in I therefore I is the mean of A and D .

Thus the element of intersection of the two diagonal lines BC and AD is the mean both of B and C and also of A and D .

Again, since CD and FH intersect in K , therefore K is the mean of C and D , while similarly J is the mean of A and B .

Further K and J both lie in a general line through the element I parallel to AC and BD .

Thus the first part of the theorem is proved provided C and G are not identical as is also the second part if the general line through the element of intersection of the diagonal lines be taken parallel to the optical side lines.

As regards the exceptional case in which C and G coincide we also have H and B coincident with one another and also with L , N and M .

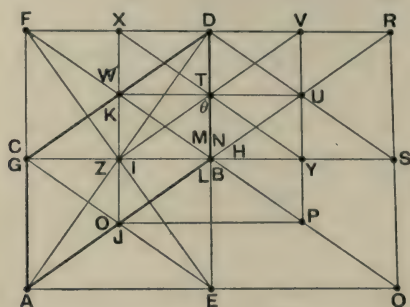


Fig. 20.

The diagonal line CB becomes an optical line and so the mean of C and B is the element in which CB is intersected by JK .

Also since B coincides with H and with L the optical line CB passes through L .

But as before the mean of A and D lies both in the optical line JK and an optical line through L parallel to FR .

Thus as before the mean of C and B coincides with the mean of A and D in the element I , while K the mean of C and D and J the mean of A and B both lie in the optical line through I parallel to AC and BD .

Thus the theorem holds in this exceptional case as regards part (1) and also as regards part (2), when the general line through the element of intersection of the diagonal lines is taken parallel to the optical side lines.

It remains to prove the theorem for the case of a general line taken through I parallel to AB and CD .

Taking first the case in which C and G do not coincide, let GB intersect QR in S .

Then D, B, S, R are the corners of an optical parallelogram while F, A, Q, R are the corners of another optical parallelogram having the diagonal line AR in common with it.

Thus the other diagonal lines DS and FQ do not intersect.

Let DS and BR intersect in U and let an optical line through U parallel to FR intersect the optical line BD in θ and FQ in W .

Then θ is the mean of B and D by definition.

Let an optical line through W parallel to AC intersect AR in O , CN in Z and FR in X .

Let an optical line through U parallel to AC intersect FQ in P , CN in Y and FR in V .

Let an optical line through O parallel to WU intersect UV in P' .

Then W, O, P', U are the corners of an optical parallelogram having the diagonal line OU in common with the optical parallelogram whose corners are F, A, Q, R .

Thus the other diagonal lines cannot intersect.

But the corner W lies in the diagonal line FQ and so P' must also lie in FQ .

Thus P' is identical with P , and WP and OU intersect in L the centre of the optical parallelogram whose corners are F, A, Q, R .

Now D, θ, U, V form the corners of an optical parallelogram whose diagonal line DU does not intersect the diagonal line FQ of the optical parallelogram whose corners are F, A, Q, R .

Thus the diagonal lines θV and AR do not intersect and therefore are parallel.

Now provided U does not coincide with N the elements θ and W will both be distinct from H and from one another.

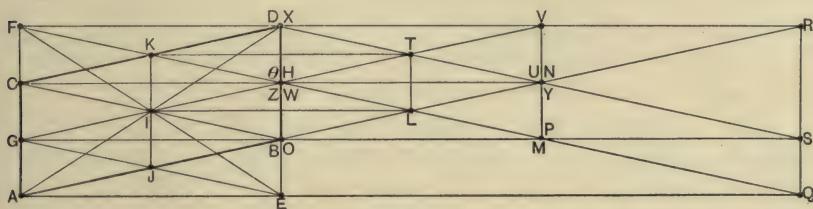


Fig. 21.

If however U coincides with N the elements θ, H, W will all coincide with one another and also with the element Z .

Further, in this case X coincides with D , O with B , Y with U and N and finally P with M .

If we suppose U is distinct from N then H, θ, W, Z form the corners of an optical parallelogram whose diagonal line HW does not intersect the diagonal line DU of the optical parallelogram whose corners are D, θ, U, V .

Thus their other diagonal lines θZ and θV do not intersect and since they have the element θ in common they must be identical.

Thus whether U coincides with N or not, the general lines ZV and θV are identical.

Now X, Z, Y, V form the corners of an optical parallelogram.

Let the diagonal lines XY and ZV intersect in T , then T is the

centre of the optical parallelogram and lies in the general line ZV or θV .

But now the optical parallelograms whose corners are X, Z, Y, V and W, O, P, U have a pair of opposite side lines in common and so their centres T and L lie in an optical line parallel to OX and PV .

Also the optical parallelograms whose corners are X, Z, Y, V and F, C, H, D have a pair of opposite side lines in common and so their centres T and K lie in an optical line parallel to FV .

But K, I, L, T form the corners of an optical parallelogram having the diagonal line KL which does not intersect the diagonal line DU of the optical parallelogram whose corners are D, θ, U, V .

Thus their other diagonal lines IT and θV do not intersect.

But the general lines IT and θV have the element T in common and so must be identical.

Thus IT is parallel to AR : that is, to AB and CD ; and passes through θ the mean of B and D .

Similarly it must pass through the mean of A and C .

Thus the second part of the theorem is proved also in this case, provided C and G are not identical.

There is no special difficulty in proving it for this case also.

As we have already seen CB and AD intersect in the element I which is the mean both of C and B and of A and D .

Also we have seen that an optical line through I parallel to AC and BD intersects CD in the element K which is the centre of the optical parallelogram whose corners are F, C, H, D .

Now CB is in this case an optical line and so, if we take an optical line through K parallel to CB and intersecting BD in θ , we shall have K, I, B, θ the corners of an optical parallelogram having the diagonal line KB in common with the optical parallelogram whose corners are F, C, H, D .

Thus the other diagonal lines $I\theta$ and GD do not intersect and therefore are parallel.

But by definition, θ is the mean of B and D and so in this case also a general line through the element I of intersection of the diagonal lines CB and AD parallel to CD intersects BD in an element which is the mean of B and D .

Similarly it intersects AC in an element which is the mean of A and C .

Thus the theorem holds in this case also.

We have thus proved the theorem to hold for the case of a general parallelogram in an acceleration plane having a pair of opposite side lines optical lines.

It remains to prove it without this latter restriction.

Let A, B, C, D be the corners of a general parallelogram in an acceleration plane and let AC and BD be one pair of opposite side lines while AB and CD are the other pair of opposite side lines.

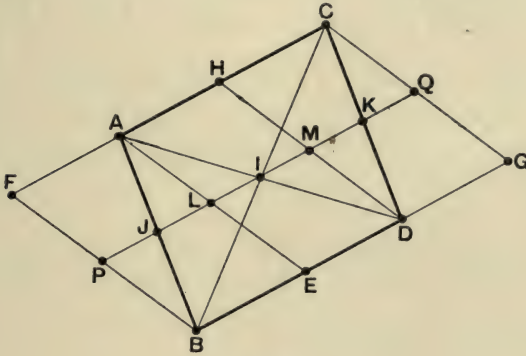


Fig. 22.

We shall suppose that neither pair of opposite side lines are optical lines, since we have already proved the theorem for that case.

Let AE and CG be parallel optical lines in the acceleration plane through the elements A and C and intersecting the general line BD in E and G respectively.

Let other optical lines parallel to AE and CG through the elements B and D intersect the general line AC in F and H respectively.

Then by the case already proved a general line through the mean of B and C parallel to BD will intersect BF in the mean of B and F , say P , and will intersect GC in the mean of G and C , say Q .

Similarly a general line through the mean of D and C parallel to BD will intersect GC in the mean of G and C : that is, in Q ; and will intersect DH in the mean of D and H , say M .

Since there is only one parallel to BD through Q , therefore M must lie in PQ .

In a similar manner a general line through the mean of B and A parallel to BD will intersect BF in the mean of B and F : that is, in P ; and will intersect EA in the mean of E and A , say L .

Then L must also lie in PQ .

Let J be the mean of B and A , while K is the mean of D and C .

Again a general line through the mean of D and A parallel to BD will intersect EA in the mean of E and A : that is, in L ; and will intersect DH in the mean of D and H : that is, in M .

Thus the mean of B and C and also the mean of D and A both lie in the general line JK which passes through the mean of B and A and the mean of D and C and is parallel to BD .

Similarly it may be proved that the mean of B and C and also the mean of D and A both lie in a general line which passes through the mean of A and C and also through the mean of B and D and is parallel to AB .

Thus since the mean of B and C and the mean of D and A both lie in two distinct general lines these means must be identical and accordingly the two diagonal lines BC and DA must intersect in an element, say I , such that I is the mean of B and C and also the mean of D and A .

Further a general line through I parallel to either pair of side lines intersects either of the other side lines in an element which is the mean of the pair of corners through which that side line passes.

Thus the theorem holds in general.

THEOREM 81.

If A , B and C be three elements in an acceleration plane which do not all lie in one general line and if D be the mean of A and B , then:

(1) *A general line through D parallel to BC intersects AC in an element which is the mean of A and C .*

(2) *If E be the mean of A and C the general line DE is parallel to BC .*

Let a general line be taken through A parallel to BC and let a general line be taken through C parallel to BA and let the two general lines intersect in F .

Then A , B , C , F are the corners of a general parallelogram in the acceleration plane and A and C are a pair of opposite corners.

Then, by Theorem 80, a general line through the mean of A and C parallel to BC passes through the mean of A and B . That is, through D .

Thus, since there is only one general line through D parallel to BC , the parallel to BC through D must intersect AC in an element which is the mean of A and C .

Again, if E be the mean of A and C there is only one general line passing through both D and E and accordingly this must be identical with the parallel to BC through either of these elements.

Thus both parts of the theorem are proved.

THEOREM 82.

If three parallel general lines a , b and c in one acceleration plane intersect a general line d_1 in A_1 , B_1 and C_1 respectively and intersect a second general line d_2 in A_2 , B_2 and C_2 respectively, and if B_1 be the mean of A_1 and C_1 , then B_2 will be the mean of A_2 and C_2 .

If A_2 should happen to coincide with A_1 , or if C_2 should happen to coincide with C_1 , the result follows directly from Theorem 81.

If d_2 should happen to be parallel to d_1 , then the result follows from Theorem 80 (2).

In any other case let a general line through A_1 parallel to d_2 intersect b in B and c in C .

Then, by Theorem 81, B is the mean of A_1 and C and so, by Theorem 80 (2), B_2 will be the mean of A_2 and C_2 .

REMARKS.

If A_0 and A_n be two distinct elements in a general line a , we can always find $n-1$ elements A_1, A_2, \dots, A_{n-1} in a (where $n-1$ is any integer) such that:

A_1 is the mean of A_0 and A_2 ,
 A_2 is the mean of A_1 and A_3 ,

 A_{n-1} is the mean of A_{n-2} and A_n .

For let P be any acceleration plane containing a and let b be any general line in P which passes through A_0 and is distinct from a .

Let A'_1 be any element in b distinct from A_0 and let $A'_2, A'_3, \dots, A'_{n-1}, A'_n$ be other elements in b such that:

A'_1 is the mean of A_0 and A'_2 ,
 A'_2 is the mean of A'_1 and A'_3 ,

 A'_{n-1} is the mean of A'_{n-2} and A'_n .

Let general lines through $A'_1, A'_2, \dots, A'_{n-1}$ parallel to $A'_n A_n$ intersect a in the elements A_1, A_2, \dots, A_{n-1} .

Then, by Theorem 82, it follows that:

A_1 is the mean of A_0 and A_2 ,
 A_2 is the mean of A_1 and A_3 ,

 A_{n-1} is the mean of A_{n-2} and A_n ,

and so the $n-1$ elements A_1, A_2, \dots, A_{n-1} can be found as stated.

THEOREM 83.

(a) If A be any element in an optical line a and A' be any element in a neutral-parallel optical line a' , then, if B be any element in a which is after A , the general line through B parallel to AA' intersects a' in an element which is after A' .

Since A and A' lie in the neutral-parallel optical lines a and a' respectively, it follows that A is neither *before* nor *after* A' and so there is at least one element which is common to the α sub-sets of A and A' .

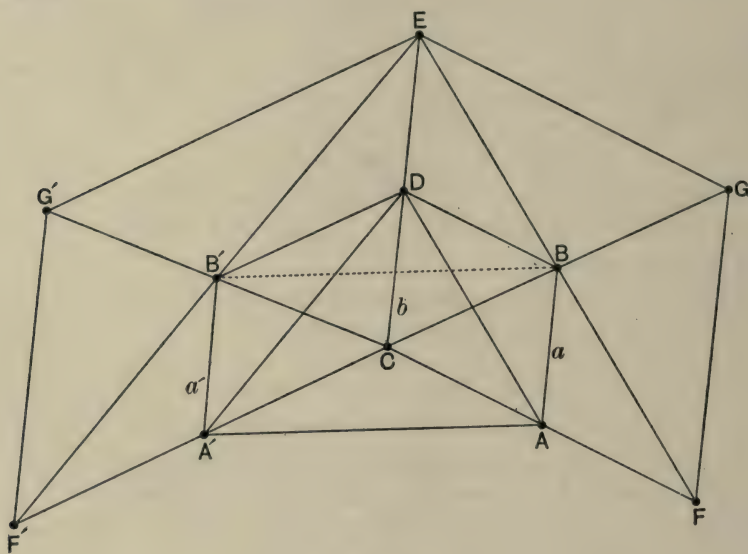


Fig. 23.

Let C be such an element and let b be the optical line through C parallel to a or a' .

Then since C is *after* both A and A' , it follows that b is an after-parallel of both a and a' and accordingly b and a lie in one acceleration plane while b and a' lie in another.

Let the optical line through B parallel to AC intersect b in the element D and let the optical line through D parallel to CA' intersect a' in the element B' .

Then A, C, D, B form the corners of an optical parallelogram in an acceleration plane which we shall call P , while A', C, D, B' form the corners of another optical parallelogram in another acceleration plane which we shall call P' .

Now since B is *after* A and C is also *after* A , while AC and AB are

both optical lines, it follows that the diagonal line CB is a separation line and accordingly the diagonal line AD is an inertia line having D after A .

Further D must be after C and, since C is after A' , it follows that the diagonal line $A'D$ is an inertia line having D after A' , and accordingly the diagonal line CB' is a separation line.

Thus since C is after A' , we must also have B' after A' .

Let the general line through B parallel to AD intersect b in E and CA in F , and let the optical lines through E and F respectively parallel to CF and CE intersect one another in G .

Then F, C, E, G are the corners of an optical parallelogram in the same acceleration plane as the optical parallelogram whose corners are A, C, D, B and the diagonal lines FE and AD do not intersect and so the diagonal lines CG and CB do not intersect.

Thus B must lie in CG and since it also lies in FE it follows that B is the centre of the optical parallelogram whose corners are F, C, E, G .

Now let the general line through E parallel to DA' intersect CA' in F' and let the optical lines through E and F' respectively parallel to CF' and CE intersect one another in G' .

Then F', C, E, G' are the corners of an optical parallelogram in the same acceleration plane as the optical parallelogram whose corners are A', C, D, B' and the diagonal lines $F'E$ and $A'D$ do not intersect and so the diagonal lines CG' and CB' do not intersect.

Thus B' lies in CG' .

But the optical parallelograms whose corners are F, C, E, G and F', C, E, G' have the pair of adjacent corners C and E in common and the optical line BD through the centre of the first of these intersects CE in D and so it follows by Theorem 60 that the centre of the second optical parallelogram lies in the optical line through D parallel to CF' and EG' .

Thus the centre of the optical parallelogram whose corners are F', C, E, G' lies in DB' .

But this centre also lies in CG' and therefore it must be the element B' .

Thus B' must lie in $F'E$.

But we saw that AD and $A'D$ were both inertia lines and so they lie in an acceleration plane, say Q_1 , while BE and $B'E$ which are respectively parallel to these must lie in a parallel acceleration plane, say Q_2 .

Further AC and $A'C$ are both optical lines and so they lie in an

Let any other general line through A' and intersecting b intersect it in the element C .

Then if C should coincide with B the general lines $A'C$ and AB have the element B in common and therefore intersect.

Suppose next that C does not coincide with B .

Let P_1 be any acceleration plane containing a and let P_2 be the parallel acceleration plane containing b .

Let a_1 be any inertia line through A in the acceleration plane P_1 and let Q be the acceleration plane containing a_1 and the element B .

Then Q must contain a general line, say b_1 , in common with P_2 and the general lines a_1 and b_1 must be parallel.

Again let a_1' be a general line through A' parallel to a_1 .

Then a_1' must lie in the acceleration plane P_1 and must be an inertia line.

Thus the general line a_1' and the element B' must lie in an acceleration plane, say Q' , and since a_1' is parallel to a_1 and $A'B'$ is parallel to AB , it follows by Theorem 52 that Q' is parallel to Q .

But the acceleration plane Q' contains the general line a_1' in P_1 and the element B' in P_2 and therefore since P_1 and P_2 are parallel it follows that Q' and P_2 contain a general line, say b_1' , in common, which will be parallel to a_1' .

Again, since a_1' is an inertia line, there is an acceleration plane containing a_1' and the element C .

If we call this acceleration plane R , then by Theorem 51 the acceleration planes P_2 and R have a general line, say c_1 , in common and c_1 is parallel to a_1' and b_1' .

Thus since c_1 lies in P_2 and R , b_1' in Q' and P_2 , and a_1' in R and Q' , and since Q is an acceleration plane parallel to Q' through the element B of P_2 which does not lie in b_1' , it follows by Theorem 53 that the acceleration planes R and Q have a general line in common, say f_1 , which is parallel to a_1' .

Now since C is neither *before* nor *after* A' , it follows that $A'C$ is a separation line and therefore must intersect the inertia line f_1 since both lie in one acceleration plane R .

Similarly AB is a separation line and must intersect the inertia line f_1 since both lie in the acceleration plane Q .

Let AB intersect f_1 in F and let $A'C$ intersect f_1 in F' .

We have to show that F' is identical with F .

Let f be the optical line through F parallel to a and let f' be the optical line through F' parallel to a .

Then since B is neither *before* nor *after* any element of a , it follows

by Theorem 45 that no element of the general line AB with the exception of A is either *before* or *after* any element of a ; and similarly no element of the general line $A'C$ with the exception of A' is either *before* or *after* any element of a .

But F cannot be identical with A , for this would require C to lie in P_1 , which is impossible, and F' cannot be identical with A' since F' and A' lie in parallel acceleration planes Q and Q' .

Thus F is neither *before* nor *after* any element of a and F' is neither *before* nor *after* any element of a .

It follows that f is a neutral-parallel of a and also f' is a neutral-parallel of a .

Suppose now, if possible, that F' is distinct from F ; then since F and F' lie in the inertia line f_1 , it would follow that the one was *after* the other.

Also if they were distinct, since they both lie in the same inertia line they could not also lie in one optical line and so the optical lines f and f' would be distinct and the one would be an after-parallel of the other.

But we have seen that f and f' are each neutral-parallels of a and so it would follow by Theorem 27 that they were neutrally parallel to one another.

But one optical line cannot be both a neutral-parallel and an after-parallel of another optical line and so the supposition that F' is distinct from F leads to a contradiction and therefore is not true.

Thus F' is identical with F and therefore the general line $A'C$ intersects the general line AB .

Thus there is no general line through A' and intersecting b which does not also intersect AB , except the parallel general line $A'B'$.

THEOREM 85.

If a and b be two neutral-parallel optical lines and if one general line intersects a in A and b in B , while a second general line intersects a in A' and b in B' , then an optical line through any element of AB and parallel to a or b intersects $A'B'$.

Let D be any element of AB and let d be an optical line through D parallel to a or b .

We have to show that d intersects $A'B'$.

If D should coincide with either A or B , no proof is required.

If $A'B'$ be parallel to AB , then the result follows directly by Theorem 83 (a) and (b).

If $A'B'$ be not parallel to AB , then by Theorem 84 the general lines AB and $A'B'$ must intersect in some element, say C .

Now, the general lines AB and $A'B'$ being supposed distinct, C must be distinct from at least one of the elements A and B and without limitation of generality we may suppose that C is distinct from B .

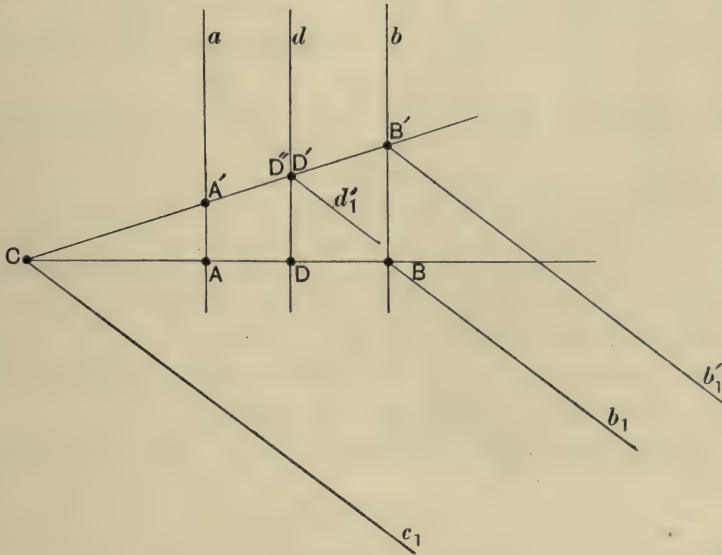


Fig. 25.

Let Q be any acceleration plane containing the optical line b and let b_1 be any inertia line through B in Q .

Let b_1' be the parallel inertia line through B' which will also lie in Q .

Let P be the acceleration plane containing b_1 and C , while R is the acceleration plane containing b_1' and C .

Then by Theorem 51 P and R have a general line, say c_1 , in common, which is parallel to b_1 and b_1' .

Suppose that D is not identical with B and let Q' be the acceleration plane through D and parallel to Q .

Then we have the three distinct acceleration planes P , Q and R and the three parallel general lines c_1 , b_1 and b_1' , such that c_1 lies in P and R , b_1 in Q and P , and b_1' in R and Q , while Q' is an acceleration plane parallel to Q through an element of P which does not lie in b_1 , and so by Theorem 53 the acceleration planes R and Q' have a general line in common which is parallel to b_1' .

Call this general line d_1' .

Then d_1' is an inertia line.

Now the optical line d must lie in Q' and must therefore intersect d_1' in some element, say D' .

Also $A'B'$ being a separation line in the acceleration plane R must intersect the inertia line d_1' in some element, say D'' .

We have to show that D'' is identical with D' .

Suppose if possible that D'' is distinct from D' and let d'' be the optical line through D'' parallel to b .

Then since, by Theorem 45, D is neither *before* nor *after* any element of b , it follows that d is a neutral-parallel of b .

Similarly d'' is a neutral-parallel of b and so if D' and D'' were distinct and did not lie in the same optical line, it would follow by Theorem 27 that d'' was a neutral-parallel of d .

But D' and D'' lie in d_1' which is an inertia line and so if D' and D'' were distinct one of them would have to be *after* the other and so d and d'' could not be neutral-parallels.

Thus the supposition that D'' is distinct from D' leads to a contradiction and so D'' must be identical with D' .

Thus the optical line d intersects $A'B'$ in D' which proves the theorem.

THEOREM 86.

If a and b be two neutral-parallel optical lines and E be any element in a separation line AB which intersects a in A and b in B , and if $A'B'$ be any other separation line intersecting a in A' and b in B' but not parallel to AB , then E either lies in $A'B'$ or in a separation line parallel to $A'B'$ which intersects both a and b .

If E does not lie in $A'B'$, then by Theorem 85 an optical line through E parallel to a or b intersects $A'B'$ in some element, say E' , which is either *before* or *after* E .

Thus by Theorem 83 the general line through E parallel to $A'B'$ intersects a and similarly it intersects b .

Thus E must lie in a separation line parallel to $A'B'$ and intersecting both a and b when it does not lie in $A'B'$ itself.

REMARKS.

If a and b be two neutral-parallel optical lines and if c and d be any two non-parallel separation lines intersecting both a and b , then it is evident from Theorem 86 that: the aggregate consisting of all the elements in c and in all separation lines intersecting a and b which are

parallel to c must be identical with the aggregate consisting of all the elements in d and in all separation lines intersecting a and b which are parallel to d .

This follows since each element in the one set of separation lines must also lie in the other set.

Thus the aggregate which we obtain in this way is independent of the particular set of parallel separation lines intersecting a and b which we may select and so we have the following definition.

Definition. The aggregate of all elements of all mutually parallel separation lines which intersect two neutral-parallel optical lines will be called an *optical plane**.

It is evident that through any element of an optical plane there is *one single optical line* lying in the optical plane.

For if a and b be two neutral-parallel optical lines which are intersected by a separation line d in the elements A and B respectively, and if C be any other element in d , then there is a neutral-parallel to a and b through C which we may call c .

But through each element of c other than C there is a separation line parallel to d which, by Theorem 83 (a) and (b), must intersect both a and b , and so every element of the optical line c lies in the optical plane defined by a and b .

An optical plane differs in this respect from an acceleration plane, since the latter contains two optical lines passing through any element of it.

Definition. In analogy with the case of an acceleration plane, an optical line which lies in any optical plane will be called a *generator* of the optical plane.

THEOREM 87.

If two distinct elements of a general line lie in an optical plane, then every element of the general line lies in the optical plane.

Let the optical plane be determined by the two neutral-parallel optical lines a and b .

If the two elements lie in a general line which is known to intersect both a and b , no proof is required.

Let C be any element in any separation line AB which intersects a in A and b in B , and let D' be any element in any separation line $A'B'$ parallel to AB and intersecting a in A' and b in B' .

* The name "optical plane" has been adopted because of certain analogies with an optical line.

We have to show that every element of the general line CD' lies in the optical plane.

By Theorem 83 (a) or (b) an optical line through C parallel to a or b will intersect $A'B'$ in some element, say C' .

If C' should coincide with D' , then CD' would be an optical line which would be neutrally parallel to a or b and we already know that each element of it must lie either in a separation line parallel to AB and intersecting both a and b , or in AB itself.

Thus if C' should coincide with D' , the general line CD' is such that every element of it lies in the optical plane.

If C' does not coincide with D' , then an optical line through D' parallel to CC' will intersect AB in some element, say D (Theorem 83 (a) or (b)).

Now DD' must be a neutral-parallel of CC' and either of the optical lines a or b must be either parallel to CC' and DD' or identical with one of them.

If a is identical with CC' or DD' , then a intersects CD' , while if b is identical with CC' or DD' then b intersects CD' .

If a is not identical with CC' or DD' , then, by Theorem 85, a must intersect CD' and similarly if b is not identical with CC' or DD' then b must intersect CD' .

Thus in all these cases CD' intersects both a and b and therefore every element of CD' lies in the optical plane determined by a and b .

THEOREM 88.

If e be a general line in an optical plane and A be any element of the optical plane which does not lie in e , then there is one single general line through A in the optical plane which does not intersect e .

We saw in the course of proving Theorem 87 that if an optical plane be determined by two neutral-parallel optical lines a and b , then any general line containing two elements in the optical plane and therefore any general line lying in the optical plane, must either be a neutral-parallel of a or b , or else must intersect both a and b .

Suppose first that e is a separation line in the optical plane determined by a and b , then e must intersect both a and b .

Since A does not lie in e it must lie in a separation line f parallel to e and intersecting both a and b .

Now through A there is an optical line, say c , which is a neutral-parallel of a or b and which by Theorem 83 (a) and (b) must intersect e and must lie in the optical plane, while any other general line f' through A and lying in the optical plane must intersect both a and b .

But f' is not parallel to e and therefore by Theorem 84 it must intersect it.

Suppose next that e is an optical line.

Then e must either be parallel to a and b or be identical with one of them.

Through A there is an optical line parallel to a or b and therefore parallel to e , and this optical line must lie in the optical plane.

Any other general line through A in the optical plane intersects both a and b and so by Theorem 85 it must also intersect e .

Thus there is in all cases one single general line through A in the optical plane which does not intersect e .

THEOREM 89.

If A , B and C be three elements in an optical plane which do not all lie in one general line and if D be an element linearly between A and B , while E is an element linearly between B and C , there exists an element which lies both linearly between A and E and linearly between C and D .

Let a_1 be any inertia line through A while b_1 and c_1 are parallel inertia lines through B and C respectively.

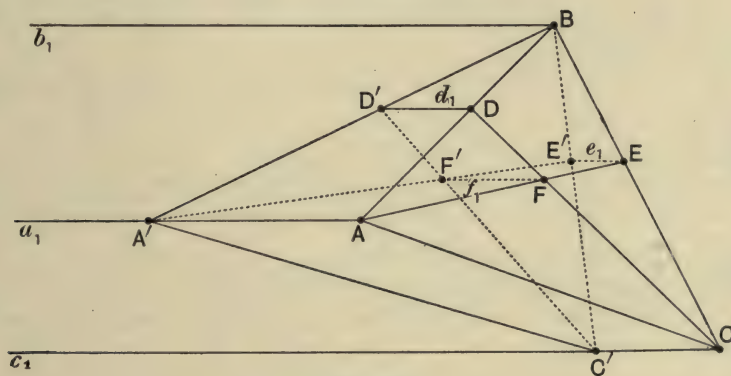


Fig. 26.

Then b_1 and c_1 lie in one acceleration plane, say P , c_1 and a_1 in a second acceleration plane, say Q , and a_1 and b_1 in a third acceleration plane, say R .

Let one of the optical lines through B in the acceleration plane P intersect c_1 in C' and let one of the optical lines through B in the acceleration plane R intersect a_1 in A' .

Then BC' and BA' may be taken as generators of opposite sets of an acceleration plane, say S , containing B , C' and A' .

Let d_1 be an inertia line through D parallel to b_1 and let e_1 be an inertia line through E parallel to b_1 .

Then d_1 will lie in R and since D is linearly between A and B it follows by Theorem 78 that d_1 must intersect $A'B$ in some element, say D' , such that D' is linearly between A' and B .

Similarly e_1 will lie in P and since E is linearly between B and C it follows that e_1 will intersect BC' in some element, say E' , such that E' is linearly between B and C' .

But since A' , B , C' are three elements in the acceleration plane S and do not all lie in one general line, it follows by Theorem 76 that there exists an element, say F' , which lies both linearly between A' and E' and linearly between C' and D' .

Now since a_1 and c_1 are parallel inertia lines lying in the acceleration plane Q it follows that there is an acceleration plane, say T , containing a_1 and the element F' , and similarly there is an acceleration plane, say U , containing c_1 and the element F' .

Thus by Theorem 51 the acceleration planes T and U have a general line, say f_1 , in common, which is parallel to a_1 and c_1 and is therefore an inertia line.

But the acceleration plane T contains the general line $A'F'$ and must therefore contain E' and the inertia line e_1 which passes through E' and is parallel to a_1 .

Thus T contains the element E and therefore contains the general line AE .

Similarly U contains the general line CD .

But since F' is linearly between A' and E' it follows by Theorem 79 that the inertia line f_1 must intersect AE in some element, say F , such that F is linearly between A and E .

Similarly since F' is linearly between C' and D' it follows that f_1 must intersect CD in some element, say \bar{F} , such that \bar{F} is linearly between C and D .

But both F and \bar{F} must lie in the optical plane through A , B and C and, if distinct, could therefore only lie in an optical line or a separation line.

But F and \bar{F} each lie in the inertia line f_1 and so it follows that they cannot be distinct.

Thus the element F is both linearly between A and E and linearly between C and D .

It may happen in this and the following theorem that C' coincides with C , or A' with A , but this does not affect the validity of the method of proof.

THEOREM 90.

If A , B and C be three elements in an optical plane which do not all lie in one general line and if D be an element linearly between A and B while F is an element linearly between C and D , there exists an element, say E , which is linearly between B and C and such that F is linearly between A and E .

Let a_1 be any inertia line through A while b_1 and c_1 are parallel inertia lines through B and C respectively.

Let P , Q , R , C' , A' , S , d_1 , D' have the same significance as in the last theorem, and let U be the acceleration plane containing the parallel inertia lines c_1 and d_1 .

Let f_1 be an inertia line through F parallel to c_1 and d_1 and which will also lie in U .

Since F is linearly between C and D it follows by Theorem 79 that f_1 will intersect $C'D'$ in some element, say F' , such that F' is linearly between C' and D' .

But as in the last theorem D' is linearly between A' and B and so, since A' , B , C' lie in the acceleration plane S , it follows by Theorem 77 that there exists an element, say E' , which is linearly between B and C' and such that F' is linearly between A' and E' .

If now we denote the acceleration plane containing a_1 and f_1 by T , then T contains the element E' in common with the acceleration plane P .

But since a_1 lies in R and T while the parallel inertia line b_1 lies in R and P , it follows by Theorem 51 that P and T have a general line, say e_1 , in common, which is parallel to a_1 and b_1 and is therefore an inertia line.

Now since c_1 must also be parallel to e_1 and lies in the same acceleration plane P with it and since E' is linearly between B and C' , it follows, by Theorem 78, that e_1 must intersect BC in some element, say E , such that E is linearly between B and C .

Again, since a_1 , f_1 and e_1 all lie in the acceleration plane T and since F' is linearly between A' and E' , it follows by Theorem 79 that AF must intersect e_1 in some element \bar{E} such that F is linearly between A and \bar{E} .

But E and \bar{E} must each lie in the optical plane through A , B and C and so E and \bar{E} , if distinct, can only lie in an optical or separation line.

But E and \bar{E} each lie in the inertia line e_1 and so E and \bar{E} cannot be distinct.

Thus the element E is linearly between B and C and the element F is linearly between A and E .

REMARKS.

It will be observed that Theorem 89 is the analogue of Peano's axiom (14) for the case of elements in an optical plane, while Theorem 90 is the corresponding analogue of his axiom (13).

Further, Theorem 88 corresponds to the Euclidean axiom of parallels for the case of general lines in an optical plane.

THEOREM 91.

If two elements A and B lie in one optical line and if two other elements C and D lie in a neutral-parallel optical line, and if A be after B , then :

(1) *If C be after D the general lines AD and BC intersect in an element which is both linearly between A and D and linearly between B and C .*

(2) *If the general lines AD and BC intersect in an element E which is either linearly between A and D or linearly between B and C , we shall also have C after D .*

Let A and B lie in an optical line a and let C and D lie in a neutral-parallel optical line c .

Let a_1 be any inertia line through A and let b_1 be a parallel inertia line through B .

Then a_1 and b_1 lie in an acceleration plane, say P .

Let B' be any element in b_1 which is *after* B and let a' be an optical line through B' parallel to a .

Then a' will intersect a_1 in some element, say A' , and, by Theorem 56, since A is *after* B , we must have A' *after* B' .

But since B' is not an element of a but is *after* B an element of a , it follows that a' is an after-parallel of a .

Since further a and c are neutral-parallels, it follows by Theorem 25 (b) that a' is an after-parallel of c .

Thus a' and c lie in an acceleration plane, say Q .

Proceeding now to prove the first part of the theorem, we have A' *after* B' and C *after* D and so it follows by Theorem 67 and the definition of "linearly between" that $A'D$ and $B'C$ intersect in an element, say E' , which is linearly between A' and D and also linearly between B' and C .

But since a_1 is an inertia line there is an acceleration plane containing a_1 and the element E' which we may call R , and similarly there is an acceleration plane containing b_1 and the element E' which we may call S .

Now since a_1 and b_1 are parallel general lines in the acceleration plane P it follows, by Theorem 51, that the acceleration planes R and S have a general line, say e_1 , in common, which is parallel to a_1 and b_1 and must therefore be an inertia line.

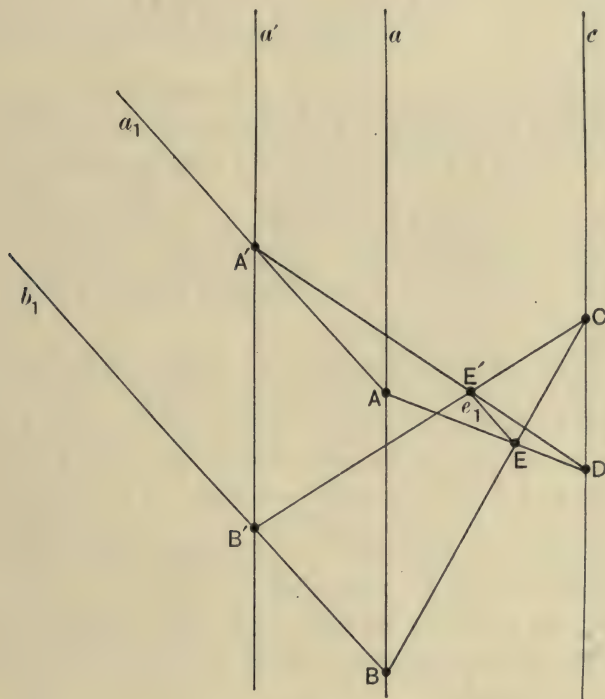


Fig. 27.

Since e_1 lies both in R and S it must intersect BC and AD which lie respectively in S and R and are separation lines.

Suppose e_1 intersects BC in E and AD in \bar{E} , then E and \bar{E} lie in the inertia line e_1 and so, if they were distinct, they could not lie in one optical plane.

But E and \bar{E} each lie in the optical plane determined by the neutral-parallel optical lines a and c and so \bar{E} is identical with E .

But since D , A and A' are elements in the acceleration plane R , which do not all lie in one general line, and since E' is linearly between

A' and D and $E'E$ is parallel to $A'A$, it follows, by Theorem 78, that E is linearly between A and D .

Similarly since E' is linearly between B' and C , and C , B and B' lie in the acceleration plane S and are not all in one general line and since $E'E$ is parallel to $B'B$ it follows that E is linearly between B and C .

Thus the first part of the theorem is proved.

Proceeding now to prove the second part of the theorem; since AD and BC intersect in the element E and since a_1 and b_1 are inertia lines it follows that there is an acceleration plane, say R , which contains a_1 and the element E , and another acceleration plane, say S , containing b_1 and the element E .

It follows, since a_1 and b_1 are parallel and lie in the acceleration plane P , that R and S have a general line, say e_1 , in common, which is parallel to a_1 and b_1 (Theorem 51) and must therefore be an inertia line.

Now the element E of the inertia line e_1 lies in the general lines AD and BC which intersect the neutral-parallel optical lines a and c , and E is not identical with C or D and so E is neither *before* nor *after* any element of the optical line c .

Thus E cannot lie in the acceleration plane Q and so e_1 cannot have more than one element in common with Q .

If now E be linearly between A and D , then since D , A and A' lie in the acceleration plane R and are not all in one general line it follows since e_1 is parallel to AA' that e_1 must intersect $A'D$ in an element, say E' , which is linearly between A' and D .

Also since $B'C$ is not parallel to e_1 and lies in the acceleration plane S with it, it follows that $B'C$ must intersect e_1 .

But $B'C$ lies in Q and we have seen that e_1 and Q cannot have more than one element in common and therefore $A'D$ and $B'C$ intersect e_1 in the same element E' .

If we suppose instead that E is linearly between B and C , we find in a similar way that $B'C$ and $A'D$ intersect e_1 in an element E' which is linearly between B' and C .

But by the definition of "linearly between" the element E' must in either case be between the parallel optical lines a' and c in the acceleration plane Q .

Thus since a' and c are parallel optical lines in the acceleration plane Q and since A' is *after* B' and the element of intersection of $A'D$ and $B'C$ lies between a' and c it follows by Theorem 68 that C is *after* D , as was to be proved.

Then e_1 will lie in the acceleration plane Q and will intersect the optical line AC' in some element, say E' .

Now d_1 and e_1 being parallel inertia lines will lie in an acceleration plane, say T , which contains the two intersecting general lines DE and d_1 which are respectively parallel to BC and b_1 in P .

Thus by Theorem 52 the acceleration planes T and P are parallel.

But the acceleration plane S has the general line $D'E'$ in common with T and the general line $B'C'$ in common with P and so $D'E'$ is parallel to $B'C'$.

Now since A , B and B' lie in the acceleration plane R and since D is the mean of A and B and since DD' is parallel to BB' , it follows by Theorem 81, provided that A , B and B' do not lie in one general line, that D' is the mean of A and B' .

The only case in which A , B and B' do lie in one general line is when B' coincides with B and then D' is identical with D so that D' is still the mean of A and B' .

Again since A , B' and C' lie in one acceleration plane S and do not all lie in one general line and since D' is the mean of A and B' , and $D'E'$ is parallel to $B'C'$, it follows by Theorem 81 that E' is the mean of A and C' .

Further since A , C and C' lie in one acceleration plane Q , since E' is the mean of A and C' and since $E'E$ is parallel to $C'C$, it follows, provided that A , C and C' do not lie in one general line, that E is the mean of A and C .

The only case in which A , C and C' do lie in one general line is when C' coincides with C and then E' coincides with E so that E is still the mean of A and C .

(It is to be noted that we cannot have both B' coinciding with B and C' with C , for then we should have two optical lines AB' and AC' passing through the same element A and lying in an optical plane, which is impossible.)

We have thus proved the first part of the theorem.

Again to prove the second part, let E be the mean of A and C ; then there is only one general line passing through both D and E and this must be identical with the parallel to BC through either of these elements.

Thus both parts of the theorem are proved.

Definition. If a pair of parallel general lines in an optical plane be intersected by another pair of parallel general lines, then the four general lines will be said to form a *general parallelogram in the optical plane*.

The terms *corner*, *side line*, *diagonal line*, *adjacent* and *opposite* will be used in a similar sense for the case of a general parallelogram in an optical plane as for one in an acceleration plane.

THEOREM 93.

If we have a general parallelogram in an optical plane, then :

(1) *The two diagonal lines intersect in an element which is the mean of either pair of opposite corners.*

(2) *A general line through the element of intersection of the diagonal lines and parallel to either pair of opposite side lines intersects either of the other side lines in an element which is the mean of the pair of corners through which that side line passes.*

Let a and b be a pair of parallel general lines in an optical plane and let a general line c intersect a in A and b in B while another general line d parallel to c intersects a in A' and b in B' .

Then A , A' , B , B' form the corners of a general parallelogram in the optical plane.

Let C be the mean of A and B and let the general line through C parallel to a and b intersect d in D .

Let E be the mean of A and A' and let the general line through E parallel to c and d intersect b in F .

Then, by Theorem 92, since CD is parallel to AA' , therefore CD intersects BA' in some element, say O , such that O is the mean of B and A' .

Also since OD is parallel to BB' , therefore OD intersects $A'B'$ in an element which is the mean of A' and B' .

Since D is this element of intersection, it follows that D is the mean of A' and B' .

Similarly EF must intersect BA' in an element which is the mean of B and A' , that is to say in the element O , while F must be the mean of B and B' .

Thus EF and CD must intersect one another in an element which is the mean of B and A' and similarly EF and CD must intersect in an element which is the mean of B' and A .

Thus B and A' have the same mean as B' and A , or the diagonal lines BA' and $B'A$ intersect in the element O which is the mean of B and A' and also the mean of B' and A .

Also the general line through O parallel to a and b intersects c in C the mean of A and B , and intersects d in D the mean of A' and B' . Similarly with the general line through O parallel to c and d .

Thus both parts of the theorem are proved.

THEOREM 94.

If A , B and C be three elements in an optical plane which do not all lie in one general line and if D be an element linearly between A and B while DE is a general line through D parallel to BC and intersecting AC in the element E , then E is linearly between A and C .

The proof of this theorem is exactly analogous to that of Theorem 78, using Theorem 89 in place of Theorem 76, and Theorem 90 in place of Theorem 77.

THEOREM 95.

If three parallel general lines a , b and c in one optical plane intersect a general line d_1 in A_1 , B_1 and C_1 respectively and intersect a second general line d_2 in A_2 , B_2 and C_2 respectively, then if B_1 is linearly between A_1 and C_1 we shall also have B_2 linearly between A_2 and C_2 .

The proof of this theorem is exactly analogous to that of Theorem 79, using Theorem 94 in place of Theorem 78.

THEOREM 96.

If three parallel general lines a , b and c in one optical plane intersect a general line d_1 in A_1 , B_1 and C_1 respectively and intersect a second general line d_2 in A_2 , B_2 and C_2 respectively, and if B_1 be the mean of A_1 and C_1 , then B_2 will be the mean of A_2 and C_2 .

The proof of this theorem is exactly analogous to that of Theorem 82, using Theorem 92 in place of Theorem 81, and Theorem 93 in place of Theorem 80.

THEOREM 97.

(a) *If A_0 and A_x be two elements in a general line a which lies in the same optical plane with another general line b which intersects a in the element C such that either A_0 is linearly between C and A_x , or A_x is linearly between C and A_0 , and if an optical line through A_0 intersects b in B_0 so that B_0 is after A_0 , then a parallel optical line through A_x will intersect b in an element which is after A_x .*

The proof of this theorem is exactly analogous to that of Theorem 70, using Theorem 91 (1) in place of Theorem 67.

There is also a (b) form of this theorem which may be proved in an analogous manner.

THEOREM 98.

If A , B , C , D be the corners of a general parallelogram in an acceleration or optical plane; AB and DC being one pair of parallel side lines and BC and AD the other pair of parallel side lines, then if

E be the mean of *A* and *B* while *F* is the mean of *D* and *C*, the general lines *AF* and *EC* are parallel to one another.

Since the general line *AF* is not parallel to *BC*, it must intersect *BC* in some element, say *G*.

Now by Theorem 80 for the case of an acceleration plane, and by Theorem 93 for the case of an optical plane, a general line through the intersection of the diagonal lines and parallel to *BC* will intersect *AB* in the mean of *A* and *B*, and will intersect *DC* in the mean of *D* and *C*.

Thus the general line *EF* is parallel to *BC*.

But since *A*, *B* and *G* are three elements in an acceleration or optical plane which do not all lie in one general line and since *E* is the mean of *A* and *B* while *EF* is parallel to *BG*, it follows by Theorems 81 and 92 that *F* is the mean of *A* and *G*.

Similarly since *FC* is parallel to *AB* it follows that *C* is the mean of *G* and *B*.

But since *E* is the mean of *B* and *A* while *C* is the mean of *B* and *G*, it follows by Theorems 81 and 92 that *EC* is parallel to *AG*: that is, *EC* is parallel to *AF*, as was to be proved.

SETS OF THREE ELEMENTS WHICH DETERMINE OPTICAL PLANES.

If *A*₁, *A*₂ and *A*₃ be three distinct elements which do not all lie in one general line and do not all lie in one acceleration plane, they either may or may not all lie in one optical plane.

In those cases in which they do lie in an optical plane they determine the optical plane containing them.

We have the following criteria by which we may say that the three elements do lie in one optical plane.

CASE I. Three elements *A*₁, *A*₂, *A*₃ lie in one optical plane if *A*₁ and *A*₂ lie in an optical line while *A*₃ is an element which is neither *before* nor *after* any element of the optical line.

This is clearly true since if *a* be the optical line containing *A*₁ and *A*₂, there is an optical line, say *b*, through *A*₃ and neutrally parallel to *a*.

These two optical lines may be taken as generators of an optical plane which will contain *A*₁, *A*₂ and *A*₃.

Now if *P* be this optical plane it is the only one which contains *A*₁, *A*₂ and *A*₃, for suppose that *A*₁, *A*₂ and *A*₃ also lie in an optical plane *P'* determined by the two generators *a'* and *b'*.

Then, since *P'* contains *A*₂ and *A*₃, it follows by Theorem 87 that *P'* contains every element of the general line *A*₂*A*₃ and since *A*₂*A*₃ is

a separation line it cannot be parallel to either a' or b' and must therefore intersect both a' and b' .

Again since P' contains A_1 and A_2 it follows that P' contains every element of A_1A_2 : that is, it contains the optical line a .

Also since P' contains A_3 it must contain the optical line through A_3 parallel to a : that is, it contains b .

Further a cannot intersect either a' or b' and so must be either parallel to both or identical with one.

Similarly b cannot intersect either a' or b' and so must be either parallel to both or identical with one.

Now every element in the optical plane P must either lie in the separation line A_2A_3 or in a separation line parallel to A_2A_3 and intersecting a and b .

But such a separation line must also intersect a' and b' and will therefore lie in the optical plane P' .

Similarly every element in the optical plane P' must either lie in the separation line A_2A_3 or in a separation line parallel to A_2A_3 and intersecting a' and b' .

But such a separation line must also intersect a and b and will therefore lie in the optical plane P .

Thus every element in P lies also in P' and every element in P' lies also in P .

Thus the optical planes P and P' are identical and so there is only one optical plane containing the three elements A_1, A_2, A_3 .

CASE II. Three elements A_1, A_2, A_3 lie in one optical plane if A_1 and A_2 lie in a separation line while A_3 is an element which is *after* one single element of A_1A_2 , or is *before* one single element of A_1A_2 .

This may be shown as follows:

Let A_3 be *after* the one single element A_4 of the separation line A_1A_2 and let A_1A_2 be denoted by c .

Then A_3A_4 cannot be an inertia line, for, if it were, we know that it would lie in an acceleration plane containing c .

Thus the three elements A_1, A_2, A_3 would lie in one acceleration plane, contrary to what was proved on page 57.

Thus A_3A_4 cannot be an inertia line and so, since A_3 is *after* A_4 , it must be an optical line.

Now A_4 must be distinct from at least one of the two elements A_1, A_2 , and without loss of generality we may suppose A_4 distinct from A_1 .

Then A_1 is neither *before* nor *after* A_4 since they are both elements of the separation line A_1A_2 .

Further A_1 cannot be *after* any element of the optical line A_3A_4 which is *after* A_4 , for then, by Post. III, A_1 would be *after* A_4 , which is impossible.

Similarly A_1 cannot be *before* any element of the optical line A_3A_4 which is *before* A_4 .

Again A_1 cannot be *before* any element of the optical line A_3A_4 which is *after* A_4 ; for if A_5 were such an element of A_3A_4 we should have A_5 *after* two distinct elements of c and so A_5 , A_1 and A_4 would lie in one acceleration plane which would also contain A_3 , contrary to what has already been shown.

Similarly A_1 cannot be *after* any element of the optical line A_3A_4 which is *before* A_4 .

Thus A_1 is neither *before* nor *after* any element of the optical line A_3A_4 and so through A_1 there is one single optical line which is neutrally parallel to A_3A_4 .

Thus these two optical lines may be taken as generators of an optical plane and, since the separation line c intersects both these optical lines and contains the elements A_1 and A_2 , it follows that the three elements A_1 , A_2 , A_3 lie in an optical plane.

Further there is only one optical plane containing A_1 , A_2 and A_3 ; for any optical plane containing A_1 and A_2 must also contain A_4 , and since, by Case I, there is only one optical plane containing A_3 , A_4 and A_1 , it follows that there is only one optical plane containing A_1 , A_2 and A_3 .

Similarly if A_3 be *before one single element* of the separation line A_1A_2 , there is one single optical plane containing the three elements A_1 , A_2 and A_3 .

THEOREM 99.

If two optical parallelograms have a pair of opposite corners in common lying in an inertia line, then their separation diagonals are such that no element of the one is either before or after any element of the other.

Let A and B be the two common opposite corners lying in the inertia line a , and let B be *after* A .

Let C and D be the two remaining corners of the one optical parallelogram which we shall suppose to lie in the acceleration plane P , and let E and F be the two remaining corners of the other optical parallelogram which we shall suppose to lie in the acceleration plane Q .

Then by Theorem 59 the two optical parallelograms have a common centre, say O , which is *after* A and *before* B .

Then the general lines CD and EF are separation lines and so their common element is neither *before* nor *after* any element of either of them.

Let CD be denoted by c and EF by e .

Now since C and E are two distinct elements in the α sub-set of A which do not lie in one optical line, it follows by Theorem 13 that C is neither *before* nor *after* E , and similarly C is neither *before* nor *after* F .

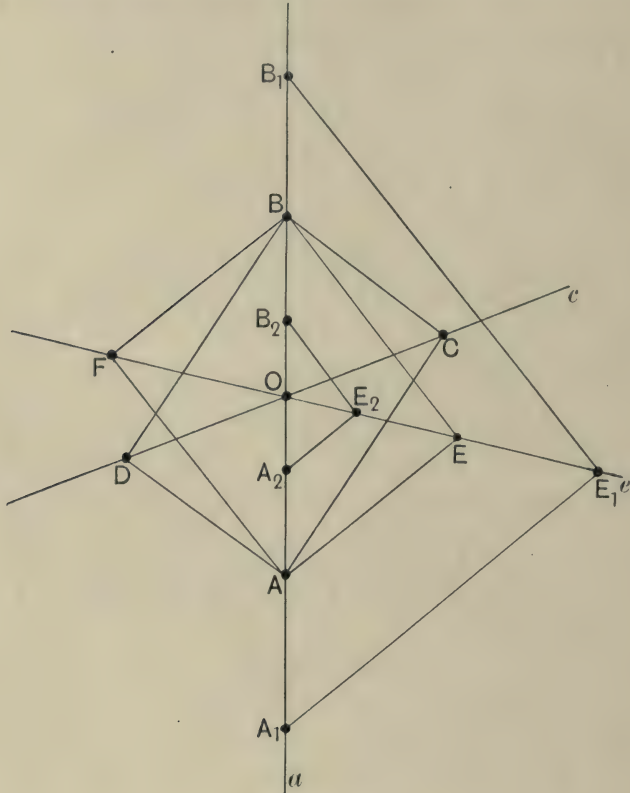


Fig. 29.

Let E_1 be any element in e such that E is linearly between O and E_1 and let the optical line through E_1 parallel to EA intersect a in A_1 , while the optical line through E_1 parallel to EB intersects a in B_1 .

Then by Theorem 78 A is linearly between A_1 and O , while B is linearly between O and B_1 .

Thus A_1 must be *before* A and B_1 must be *after* B .

Again since A_1 is *before* O , and O and E_1 lie in a separation line, we must have A_1 *before* E_1 .

Similarly since B_1 is *after* O , and O and E_1 lie in a separation line, we must have B_1 *after* E_1 .

But A is *after* A_1 and C is *after* A and therefore C is *after* A_1 and since A is the only element common to a and the β sub-set of C , it follows that A_1C is an inertia line.

If then C were *before* E_1 it would follow by Theorem 12 that C should lie in the optical line A_1E_1 which it clearly cannot do since A_1C is an inertia line.

Thus C is not *before* E_1 .

Further B is *after* C and B_1 *after* B and therefore B_1 is *after* C and B_1C is an inertia line.

If then C were *after* E_1 it would follow by Theorem 12 that C should lie in the optical line B_1E_1 which it clearly cannot do since B_1C is an inertia line.

Thus C is not *after* E_1 .

In a similar manner we may prove that C is neither *before* nor *after* any element F_1 of the separation line e such that F is linearly between O and F_1 .

Again let E_2 be any element of e which is linearly between O and E and let the optical line through E_2 parallel to EA intersect a in A_2 while the optical line through E_2 parallel to EB intersects a in B_2 .

Then we may prove in a similar manner that E_2 is neither *before* nor *after* C and therefore C is neither *before* nor *after* E_2 .

Similarly we may prove that C is neither *before* nor *after* any element F_2 of the separation line e such that F_2 is linearly between O and F .

Thus C is neither *before* nor *after* any element of the separation line e .

Similarly D is neither *before* nor *after* any element of e .

Again if C' be any other element in c distinct from O , then by Theorem 58 there is an optical parallelogram in the acceleration plane P having O as centre and C' as one of its corners.

If D' be the corner opposite to C' then D' will also lie in c , and if A' and B' be the remaining two corners these must lie in a .

Then there is one single optical parallelogram in the acceleration plane Q having A' and B' as opposite corners.

If E' and F' be the remaining corners of this optical parallelogram, then E' and F' must lie in e .

Thus we have got two new optical parallelograms having a pair of opposite corners A' and B' in common, lying in the inertia line a , while their separation diagonals are c and e respectively.

Thus we may prove in a manner similar to that already employed that C' is neither *before* nor *after* any element of e .

Thus no element of c is either *before* or *after* any element of e , as was to be proved.

REMARKS.

It is evident from the above that any general line which intersects the separation lines c and e in distinct elements must itself be a separation line.

It also appears from this theorem that it is possible to have an element which does not lie in a separation line and which is neither *before* nor *after* any element of the separation line.

If two distinct elements A_1 and A_2 lie in a separation line a , while A_3 is an element which does not lie in a and is neither *before* nor *after* any element of a , then we have already seen (page 57) that A_1 , A_2 and A_3 cannot lie in an acceleration plane and it is also evident that they cannot lie in an optical plane.

For suppose, if possible, that A_3 does lie in an optical plane containing the separation line a ; then there would be a generator of the optical plane passing through A_3 and intersecting a in some element, say A_4 .

Since A_3 is supposed not to lie in a , the elements A_3 and A_4 would require to be distinct and since they would then lie in an optical line we should have A_3 either *before* or *after* A_4 : an element of a , contrary to hypothesis.

Thus A_1 , A_2 and A_3 cannot lie in an optical plane.

Again if two distinct elements A_1 and A_2 lie in a separation line while A_3 is an element which does not lie in A_1A_2 and is neither *before* nor *after* any element of A_1A_2 , then the element A_2 is neither *before* nor *after* any element of A_3A_1 .

For if A_2 were either *before* or *after* any element of A_3A_1 , then the three elements A_1 , A_2 , A_3 would lie either in an acceleration or optical plane contrary to what we have just shown.

Similarly A_1 is neither *before* nor *after* any element of A_2A_3 .

Again if a be a separation line and A be an element which is not an element of a and is neither *before* nor *after* any element of a , then if B be any element of a , no element of the general line AB is either *before* or *after* any element of a .

This is easily seen, for suppose, if possible, that C is some element of AB which is either *before* or *after* some element of a .

Then C could not lie in a and would lie either in an acceleration or optical plane containing a .

But such acceleration or optical plane would contain the element A and so the separation line a and the element A would lie in an acceleration or optical plane, contrary to what we have already proved.

Thus no element of AB is either *before* or *after* any element of a .

Definitions. If A be any element and a be an inertia line not containing A , while B is the element common to a and the α sub-set of A , then we shall speak of B as *the first element of a which is after A* .

Similarly if C be the element common to a and the β sub-set of A , we shall speak of C as *the last element of a which is before A* .

POSTULATE XVIII. If a , b and c be three parallel inertia lines which do not all lie in one acceleration plane* and A_1 be any element in a and if

B_1 be the first element in b which is after A_1 ,

C_1 be the first element in c which is after A_1 ,

B_2 be the first element in b which is after C_1 ,

C_2 be the first element in c which is after B_1 ,

then the first element in a which is after B_2 and the first element in a which is after C_2 are identical.

It is evident that there is a (*b*) form of this postulate in which the word *last* is substituted for the word *first* and the word *before* for the word *after*, but this is not independent, as may be readily seen.

Thus let A_1 be any element in a and let B_1 be the last element in b which is *before* A_1 and let C_2 be the last element in c which is *before* B_1 while A_2 is the last element in a which is *before* C_2 .

Then C_2 is the first element in c which is *after* A_2 ,

B_1 is the first element in b which is *after* C_2 ,

A_1 is the first element in a which is *after* B_1 .

Thus if B_2 be the first element in b which is *after* A_2 and if C_1 be the first element in c which is *after* B_2 , it follows by Post. XVIII that the first element in a which is *after* C_1 is identical with the first element in a which is *after* B_1 : that is, with the element A_1 .

But C_1 is the last element in c which is *before* A_1 and B_2 is the last element in b which is *before* C_1 while A_2 is the last element in a which is *before* B_2 .

* If a , b and c do all lie in one acceleration plane, the same result may easily be deduced from the other postulates.

Thus the last element in a which is *before* B_2 and the last element in a which is *before* C_2 are identical.

Definition. An inertia line and a separation line which are diagonal lines of the same optical parallelogram will be said to be *conjugate* to one another.

It is evident that if an inertia line and a separation line are conjugate they lie in one acceleration plane and intersect one another.

It is also evident that if A be an element lying in an inertia or separation line a in an acceleration plane P , then there is only one separation or inertia line through A and lying in P which is conjugate to a ; since, if two optical parallelograms lie in P and have a as a common diagonal line, then their other diagonal lines do not intersect (Post. XVI).

From this it also follows that if two intersecting separation lines b and c be both conjugate to the same inertia line a , then a , b and c cannot lie in the same acceleration plane and we shall have a and b in one acceleration plane, say P , while a and c lie in another, say Q .

If O be the element of intersection of b and c , then O must lie both in P and Q and therefore in the inertia line a .

If A_1 be any element in a distinct from O , there is one optical parallelogram in the acceleration plane P having O as centre and A_1 as one of its corners.

If A_2 be the corner opposite A_1 , then there is an optical parallelogram in Q also having A_1 and A_2 as a pair of opposite corners and therefore having the same centre O .

The separation lines b and c will be the separation diagonals of the optical parallelograms in P and Q respectively, and so it follows by Theorem 99 that no element of b is either *before* or *after* any element of c .

By considerations similar to the above, we can see that if two intersecting inertia lines b and c be both conjugate to the same separation line a , then a and b must lie in one acceleration plane while a and c lie in another distinct acceleration plane.

Further if O be the element of intersection of b and c , then O lies in a .

In this case however, since b and c are two intersecting inertia lines, they must lie in one acceleration plane which must be distinct from both the others.

Again it is clear that if a be an inertia or separation line lying in an acceleration plane P with a separation or inertia line b which is

conjugate to a , then any general line c lying in P and parallel to b is also conjugate to a .

Also conversely it is clear that if a be an inertia or separation line lying in an acceleration plane P with two distinct separation or inertia lines b and c which are each conjugate to a , then b and c must be parallel to one another.

THEOREM 100.

If an inertia line c be conjugate to two intersecting separation lines d and e , then if A be any element of d and B be any distinct element of e , the general line AB is conjugate to a set of inertia lines which are parallel to c .

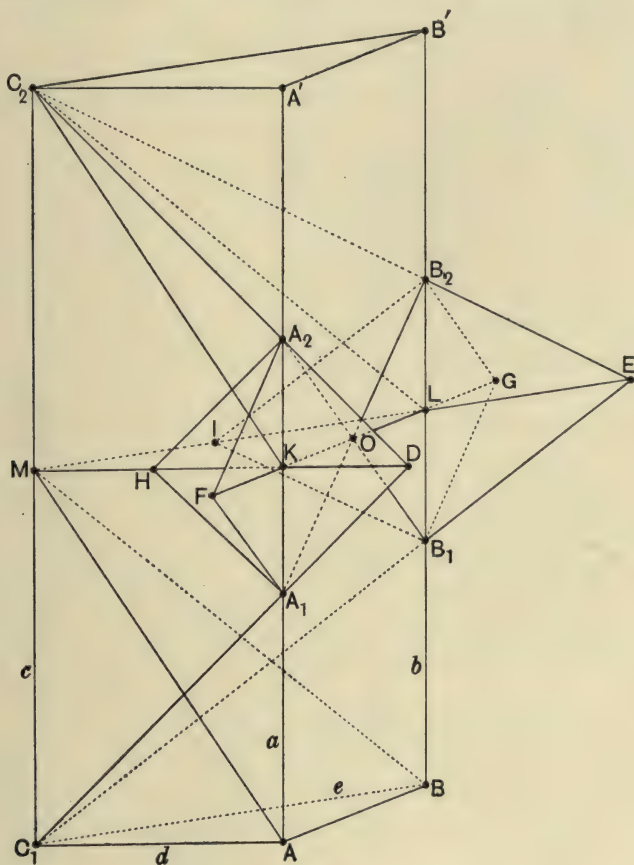


Fig. 30.

Let C_1 be the element of intersection of the separation lines d and e .

Then we know that c and d lie in one acceleration plane, say P , while c and e lie in another distinct acceleration plane, say Q , and the element C_1 lies in c .

If A or B should coincide with C_1 , then the general line AB must coincide with e or d and the result follows directly.

We shall suppose therefore that neither A nor B coincides with C_1 .

Then since by Theorem 99 A is neither *before* nor *after* B and since A and B are distinct it follows that AB is a separation line.

Let a be an inertia line through A parallel to c while b is an inertia line through B parallel to c .

Then since A and B lie in a separation line it follows that a and b must be distinct and therefore are parallel to one another.

Thus a and b must lie in an acceleration plane which we shall call R .

Further a must lie in the acceleration plane P while b must lie in the acceleration plane Q .

Now let A_1 be the first element in a which is *after* C_1 ,

let B_1 be the first element in b which is *after* C_1 ,

let A_2 be the first element in a which is *after* B_1 ,

let B_2 be the first element in b which is *after* A_1 .

If now C_2 be the first element in c which is *after* A_2 , it follows by Post. XVIII that C_2 is also the first element in c which is *after* B_2 .

Now the optical lines C_1A_1 and C_2A_2 cannot be parallel; for since A_1 is *after* C_1 and c and a are parallel inertia lines in the acceleration plane P , it would then follow by Theorem 56 (*b*) that A_2 was *after* C_2 .

But we know that C_2 is *after* A_2 and so C_1A_1 and C_2A_2 are not parallel, and, since they lie in one acceleration plane, it follows that they must intersect in some element, say D .

Similarly C_1B_1 and C_2B_2 must intersect in some element, say E .

Also since a and b are parallel inertia lines in the acceleration plane R and since B_2 is *after* A_1 and A_2 *after* B_1 , it follows in a similar manner that the optical lines A_1B_2 and A_2B_1 must intersect in some element, say O .

Now A_2 cannot be identical with A_1 for then we should have the three elements C_1 , B_1 and A_1 lying in pairs in three distinct optical lines, which is impossible by Theorem 31.

Further since B_1 is *after* C_1 and A_2 is *after* B_1 , it follows that A_2 is *after* C_1 .

But A_2 cannot be *before* A_1 for then we should have A_2 *after* one element of the optical line C_1A_1 and *before* another element of it which would entail that A_2 should lie in the optical line C_1A_1 , by Theorem 12.

We know however that A_2 and A_1 are distinct elements in the inertia line a and so A_2 cannot be *before* A_1 .

Thus since A_1 and A_2 are distinct elements in an inertia line and A_2 is not *before* A_1 , it follows that A_2 is *after* A_1 .

Similarly B_2 is *after* B_1 .

Let an optical line through A_1 parallel to DA_2 be taken and an optical line through A_2 parallel to DA_1 and let these intersect in H .

Then A_1, D, A_2, H form the corners of an optical parallelogram in the acceleration plane P , having its centre, say K , in the inertia line a .

Again let an optical line through B_1 parallel to EB_2 be taken and an optical line through B_2 parallel to EB_1 and let these intersect in I .

Then B_1, E, B_2, I form the corners of an optical parallelogram in the acceleration plane Q , having its centre, say L , in the inertia line b .

If now we take optical parallelograms having C_1 and C_2 as opposite corners in each of the acceleration planes P and Q then, by Theorem 59, these have a common centre, say M , lying in the inertia line c .

Also D will be one of the remaining corners of the optical parallelogram in P while E will be one of the remaining corners of the optical parallelogram in Q .

Thus MD and ME will each be conjugate to c .

Further since the general lines MD and d are both conjugate to c and lie in the same acceleration plane P , they must be parallel to one another.

Similarly the general lines ME and e must also be parallel to one another.

But now the optical parallelogram in the acceleration plane P having C_1 and C_2 as a pair of opposite corners, and the optical parallelogram whose corners are A_1, D, A_2, H , have diagonal lines c and a respectively which do not intersect and so since they both lie in P their other diagonal lines do not intersect.

But these other diagonal lines have the element D in common and so must be identical.

Thus the general lines MD and KD are identical and so K lies in MD .

Similarly L lies in ME .

Now let an optical line through A_1 parallel to OA_2 be taken and an optical line through A_2 parallel to OA_1 and let these intersect in F .

Then A_1, F, A_2, O form the corners of an optical parallelogram in the acceleration plane R and by Theorem 59 this must have the same centre K as the optical parallelogram whose corners are A_1, D, A_2, H .

Again let an optical line through B_1 parallel to OB_2 be taken and an optical line through B_2 parallel to OB_1 and let these intersect in G .

Then B_1, G, B_2, O form the corners of an optical parallelogram in the acceleration plane R and by Theorem 59 this must have the same centre L as the optical parallelogram whose corners are B_1, E, B_2, I .

But now the optical parallelograms whose corners are A_1, F, A_2, O and B_1, G, B_2, O have the diagonal lines a and b which do not intersect and so, since both lie in the same acceleration plane R , their other diagonal lines do not intersect.

That is, FO and GO do not intersect and so since they have the element O in common they must be identical.

Thus O lies in the general line FG ; that is, in the general line KL .

Thus KL is conjugate to both a and b .

Now let a general line through C_2 parallel to C_1A intersect a in A' , and let a general line through C_2 parallel to C_1B intersect b in B' .

Then A, A', C_1, C_2 form the corners of a general parallelogram in the acceleration plane P , while B, B', C_1, C_2 form the corners of a general parallelogram in the acceleration plane Q .

Also since MK and C_2A' are both parallel to C_1A , and since M is the mean of C_1 and C_2 , it follows by Theorem 82 that K must be the mean of A and A' .

Similarly L is the mean of B and B' .

Thus by Theorem 98 the general lines AM and KC_2 are parallel to one another and similarly the general lines BM and LC_2 are parallel to one another.

But now since A_2 is *after* A_1 and since K is the centre of optical parallelograms having A_1 and A_2 as opposite corners, it follows that K is *after* A_1 and *before* A_2 .

But since A_2 is *before* C_2 , it follows that K is *before* C_2 .

But A_2 is the only element common to a and the β sub-set of C_2 and K is distinct from A_2 .

Thus since K is *before* C_2 and does not lie in the β sub-set of C_2 , it follows that KC_2 must be an inertia line.

Similarly LC_2 is an inertia line.

Thus KC_2 and LC_2 lie in an acceleration plane, say S , while MA and MB being respectively parallel to these must, by Theorem 52, lie in an acceleration plane, say S' , parallel to S .

But now the general lines KL and AB lie in S and S' respectively and also both lie in the acceleration plane R .

Thus AB is parallel to KL and so since KL is conjugate to a and b ,

we must also have AB conjugate to a and b , and therefore also conjugate to every inertia line in R parallel to a and b .

But since a and b are parallel to c , therefore all these inertia lines are parallel to c and so the theorem is proved.

REMARKS.

If P and P' be parallel acceleration planes and if a be any generator of P , there is one single generator of P' which is a neutral-parallel of a .

This is easily seen, for if b_1 be any generator of P' belonging to the set which are not parallel to a and if B be any element in b_1 , then either:

- (1) B is *before* an element of a ,
- or (2) B is *after* an element of a ,
- or (3) B is neither *before* nor *after* any element of a .

In cases (1) and (2), since B does not lie in a and since b_1 neither intersects a nor is parallel to it, it follows by Post. XII (a and b) that there is one single element of b_1 which is neither *before* nor *after* any element of a .

Thus there is always an element of b_1 which is neither *before* nor *after* any element of a .

Let B_0 be such an element and let a' be the generator of P' parallel to a and passing through B_0 .

Then a' is a neutral-parallel of a .

Again, there can be no other generator of P' besides a' which is a neutral-parallel of a , for any other generator of P' parallel to a' must be either a *before* or *after*-parallel of a' and therefore by Theorem 25 (a or b) such a generator must be a *before* or *after*-parallel of a .

Again, if P and P' be parallel acceleration planes and if A be any element of P , while a and b are the two generators of P which pass through A , then there is one single generator of P' , say a' , which is neutrally parallel to a and there is one single generator of P' , say b' , which is neutrally parallel to b .

The optical lines a' and b' being generators of opposite sets must intersect in some element, say A' .

Then A' is neither *before* nor *after* any element of a and also neither *before* nor *after* any element of b .

Similarly A is neither *before* nor *after* any element of a' and also neither *before* nor *after* any element of b' .

The elements A and A' will be spoken of as *representatives* of one another in the parallel acceleration planes P and P' .

Thus we have the following definition.

Definition. If P and P' be parallel acceleration planes and if A and A' be elements in P and P' respectively such that the two generators of P' passing through A' are respectively neutral-parallels of the generators of P which pass through A , then the elements A and A' will be called *representatives* of one another in the parallel acceleration planes P and P' .

It is evident that the elements A and A' must lie in a separation line.

THEOREM 101.

If P_1 and P_2 be parallel acceleration planes and if A_1 be any element in P_1 while A_2 is its representative element in P_2 , then the separation line A_1A_2 is conjugate to any inertia line in P_1 which passes through A_1 and similarly A_1A_2 is conjugate to any inertia line in P_2 which passes through A_2 .

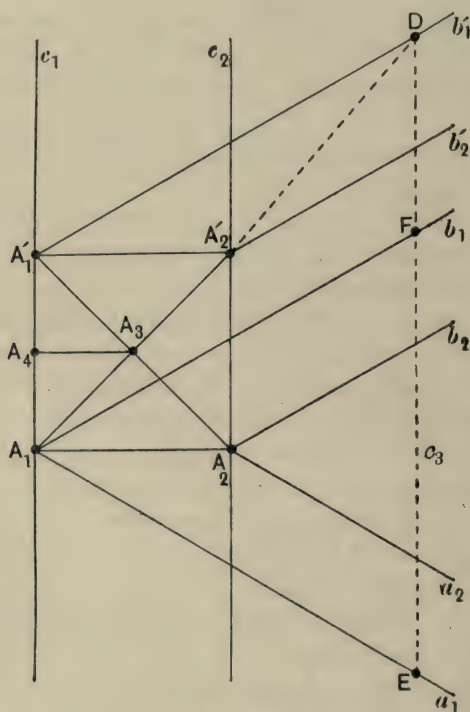


Fig. 31.

Let a_1 and b_1 be the two generators of the acceleration plane P_1 which pass through the element A_1 and let a_2 and b_2 be the two generators of the acceleration plane P_2 which pass through A_2 and let a_2 be neutrally parallel to a_1 while b_2 is neutrally parallel to b_1 .

Let c_1 be any inertia line in P_1 which passes through A_1 .

Then c_1 and the element A_2 lie in an acceleration plane, say Q , which contains the inertia line c_1 in common with P_1 and the element A_2 in common with P_2 and must therefore, by Theorem 46, have a general line in common with P_2 which must be parallel to c_1 and pass through A_2 .

Let this parallel to c_1 through the element A_2 be denoted by c_2 .

Then c_2 must also be an inertia line.

Let A_1' be the one single element common to c_1 and the α sub-set of A_2 , while A_2' is the one single element common to c_2 and the α sub-set of A_1 .

Then A_2A_1' and A_1A_2' are optical lines in the acceleration plane Q , and A_1' is *after* A_2 while A_2' is *after* A_1 .

Now A_1' cannot be either identical with A_1 or *before* A_1 for then we should have A_1 *after* A_2 , contrary to the hypothesis that A_1 and A_2 lie in a separation line.

Thus since A_1' and A_1 lie in the inertia line c_1 and since A_1' is neither identical with A_1 nor *before* A_1 , it follows that A_1' is *after* A_1 .

Similarly A_2' is *after* A_2 .

Thus the element A_1' of the optical line A_2A_1' is *after* the element A_1 of the optical line A_1A_2' while the element A_2 is *before* the element A_2' and so the optical lines A_2A_1' and A_1A_2' cannot be parallel and, since they both lie in the acceleration plane Q , they must intersect in some element, say A_3 .

Now A_1 and A_1' cannot both lie in the α sub-set of A_3 , for by Post. XIV (a) there can be only one element common to the inertia line c_1 and the α sub-set of A_3 .

Similarly by Post. XIV (b) A_1 and A_1' cannot both lie in the β sub-set of A_3 .

Thus one of the two elements A_1 and A_1' must lie in the β sub-set of A_3 while the other must lie in the α sub-set of A_3 .

But A_1' cannot lie in the β sub-set of A_3 and A_1 in the α sub-set, for then we should have A_3 *after* A_1' and A_1 *after* A_3 and therefore A_1 *after* A_1' , contrary to what we have already seen that A_1' is *after* A_1 .

Thus we must have A_1 in the β sub-set of A_3 and A_1' in the α sub-set of A_3 .

Similarly we must have A_2 in the β sub-set of A_3 and A_2' in the α sub-set of A_3 .

Now since A_1' and A_2' are two distinct elements in the α sub-set of A_3 and are both distinct from A_3 , it would follow, by Theorem 13 (a), that if A_2' were either *before* or *after* A_1' we should have A_2' in the optical line A_3A_1' .

We know however that the optical lines A_3A_1' and A_3A_2' are distinct and so A_2' is neither *before* nor *after* A_1' .

Now let b_1' be the optical line through A_1' parallel to b_1 and let b_2' be the optical line through A_2' parallel to b_2 .

Then b_1' and b_2' are parallel to one another.

We shall now show that b_2' is a neutral-parallel of b_1' , for suppose, if possible, that b_2' is a before-parallel of b_1' .

Then A_2' is not an element of b_1' , but would be *before* an element of b_1' and so there would be one single element, say D , common to the optical line b_1' and the α sub-set of A_2' .

Then D would be *after* A_2' .

But D could not be either identical with A_1' or *before* A_1' for then we should have A_1' *after* A_2' which we already know is not the case.

Thus D would require to be *after* A_1' and would therefore lie in the α sub-set of A_1' .

Now suppose c_3 to be an inertia line through D and parallel to c_1 and c_2 .

Then c_3 would lie in the acceleration plane P_1 and would intersect a_1 in some element, say E , and b_1 in some element, say F .

Now since b_1' and b_1 are parallel and since D is supposed to be *after* A_1' we should have F *after* A_1 by Theorem 56 (b).

Thus F would be in the α sub-set of A_1 , and, since there could only be one element common to the inertia line c_3 and the α sub-set of A_1 , it would follow that E must lie in the β sub-set of A_1 .

But now we should have the three parallel inertia lines c_1 , c_2 , c_3 and D an element in c_3 such that

A_1' was the last element of c_1 which was *before* D ,

A_2' was the last element of c_2 which was *before* D ,

A_1 was the last element of c_1 which was *before* A_2' ,

A_2 was the last element of c_2 which was *before* A_1' ,

E was the last element of c_3 which was *before* A_1 ,

and so E would be the last element of c_3 which was *before* A_2 .

But by hypothesis A_2 is neither *before* nor *after* any element of the optical line a_1 and therefore E could not be *before* A_2 .

Thus the supposition that b_2' is a before-parallel of b_1' leads to a contradiction and therefore is not true.

Similarly if b_2' were supposed to be an after-parallel of b_1' we should have b_1' a before-parallel of b_2' and a similar method would show this also to be impossible.

Thus since b_2' is parallel to b_1' and cannot be either a before- or

after-parallel of b_1' , it follows that b_2' must be a neutral-parallel of b_1' and the element D does not exist.

Thus, by Theorem 45, the general line $A_1'A_2'$ is such that no element of it except A_1' is either *before* or *after* any element of b_1' , and so through each element of it except A_1' there is a neutral-parallel of b_1' .

But the general line A_1A_2 is such that no element of it except A_1 is either *before* or *after* any element of b_1 and so through each element of it, except A_1 , there is a neutral-parallel of b_1 .

Now the general lines $A_1'A_2'$ and A_1A_2 cannot intersect, for if they did the element of intersection could not be identical with either A_1' or A_1 and so through such a hypothetical element of intersection there would be an optical line which would be neutrally parallel both to b_1' and to b_1 .

But if this were so it would follow by Theorem 27 that b_1' was a neutral-parallel of b_1 , contrary to what we have already seen that the element A_1' of b_1' is *after* the element A_1 of b_1 .

Thus the general lines $A_1'A_2'$ and A_1A_2 cannot intersect and so, since they both lie in the acceleration plane Q and are distinct, it follows that they are parallel.

Thus A_1, A_2, A_1', A_2' form the corners of a general parallelogram in the acceleration plane Q and the diagonal lines are A_1A_2' and A_2A_1' which are both optical lines and intersect in the element A_3 .

Thus by Theorem 80 (2) a general line through A_3 parallel to A_1A_2 will intersect A_1A_1' in some element, say A_4 , such that A_4 is the mean of A_1 and A_1' .

Thus an optical parallelogram in the acceleration plane Q having A_1 and A_1' as a pair of opposite corners will have the general line A_4A_3 as its separation diagonal line.

Thus A_4A_3 is conjugate to c_1 and since A_1A_2 is parallel to A_4A_3 and in the same acceleration plane Q with it, it follows that A_1A_2 is also conjugate to c_1 .

Similarly A_1A_2 is conjugate to any inertia line in P_2 which passes through A_2 and so the theorem is proved.

THEOREM 102.

If two inertia lines b and c intersect in an element A_1 and are both conjugate to a separation line a , then a is conjugate to every inertia line in the acceleration plane containing b and c which passes through the element A_1 .

We have already seen that a cannot lie in the acceleration plane

containing b and c and also that it passes through the element of intersection of b and c .

Let P_1 be the acceleration plane containing b and c and let A_2 be any element of a distinct from A_1 .

Let P_2 be an acceleration plane through A_2 and parallel to P_1 .

Let A_2' be the representative of A_1 in the acceleration plane P_2 .

We shall show that A_2' must be identical with A_2 .

Since the inertia line b and the separation line a intersect in the element A_1 they must lie in one acceleration plane which contains the inertia line b in common with the acceleration plane P_1 and the element A_2 in common with the parallel acceleration plane P_2 .

Thus the acceleration plane containing b and a has a general line, say b' , in common with P_2 , and b' is parallel to b and is therefore also an inertia line.

Similarly the acceleration plane containing c and a has an inertia line, say c' , in common with P_2 , and c' is parallel to c .

Further b' and c' must both pass through A_2 and must be distinct since b and c are distinct.

Now since A_1 and A_2' are representatives of one another in the parallel acceleration planes P_1 and P_2 , it follows, by Theorem 101, that the separation line A_1A_2' is conjugate to any inertia line in P_1 which passes through A_1 .

Thus A_1A_2' must be conjugate to both b and c .

Suppose now, if possible, that A_2' is distinct from A_2 .

Then b is conjugate to both A_1A_2 and A_1A_2' and so, by Theorem 100, b' would be conjugate to A_2A_2' .

Similarly c is conjugate to both A_1A_2 and A_1A_2' and so c' would be conjugate to A_2A_2' .

But then we should have two distinct inertia lines b' and c' both conjugate to the same general line A_2A_2' in the acceleration plane P_2 which contains b' and c' , and this we know is impossible.

Thus A_2' cannot be distinct from A_2 and so A_2 must be the representative of A_1 in the acceleration plane P_2 .

It follows accordingly that the separation line a is conjugate to every inertia line in P_1 which passes through A_1 , and so the theorem is proved.

It is to be noted that in proving the above theorem we have also incidentally proved the following important result:

If two inertia lines b and c intersect in an element A_1 and are both conjugate to a separation line a , then a is such that no element of it,

with the exception of A_1 , is either *before* or *after* any element of either of the generators of the acceleration plane containing b and c which pass through A_1 .

THEOREM 103.

If P_1 and P_2 be two parallel acceleration planes and if A_1 be any element in P_1 while A_2 is its representative in P_2 , then if A_1' be any other element in P_1 and A_2' its representative in P_2 the separation lines A_1A_2 and $A_1'A_2'$ are parallel to one another.

Let a_1 and b_1 be the generators of P_1 which pass through A_1 and let a_2 and b_2 be the generators of P_2 which pass through A_2 , the optical lines a_1 and a_2 being neutrally parallel to one another and the optical lines b_1 and b_2 being also neutrally parallel to one another.

Consider first the case where A_1' lies in one of the generators a_1 or b_1 which pass through A_1 .

It will be sufficient if we consider the case where A_1' lies in a_1 .

Then A_2' will lie in a_2 .

Let b_1' be the second generator of P_1 which passes through A_1' and let b_2' be the second generator of P_2 which passes through A_2' .

Then b_1' will be parallel to b_1 while b_2' will be parallel to b_2 and the optical lines b_1' and b_2' will be neutrally parallel to one another by the definition of representative elements.

Now since a_1 and a_2 are neutral-parallel optical lines they determine an optical plane which contains the separation lines A_1A_2 and $A_1'A_2'$ which must therefore either intersect or be parallel to one another.

Now, by Theorem 45, no element of the general line A_1A_2 with the exception of A_1 is either *before* or *after* any element of b_1 , and similarly, no element of the general line $A_1'A_2'$ with the exception of A_1' is either *before* or *after* any element of b_1' .

Now suppose, if possible, that A_1A_2 and $A_1'A_2'$ intersect in some element A_0 .

Then A_0 could not coincide with either A_1 or A_1' and so would require to be neither *before* nor *after* any element of b_1 and also neither *before* nor *after* any element of b_1' .

If then b_0 were an optical line through A_0 parallel to b_1 and b_1' , we should have b_0 neutrally parallel to both b_1 and b_1' .

Thus by Theorem 27 b_1 would require to be neutrally parallel to b_1' .

But b_1 and b_1' are parallel generators of the acceleration plane P_1 and so one must be an after-parallel of the other.

Thus the supposition that A_1A_2 and $A_1'A_2'$ intersect leads to a contradiction and therefore is not true.

It follows that A_1A_2 and $A_1'A_2'$ are parallel, which proves the theorem in this special case.

Next consider the case where A_1' does not lie either in a_1 or b_1 .

Let b_1' be the generator of P_1 through A_1' parallel to b_1 and let b_2' be the generator of P_2 through A_2' parallel to b_2 .

Let b_1' and a_1 intersect in B_1 and let b_2' and a_2 intersect in B_2 .

Then since a_1 and a_2 are neutrally parallel and also b_1' and b_2' are neutrally parallel, it follows by the case already proved that A_1A_2 and B_1B_2 are parallel to one another.

Similarly $A_1'A_2'$ and B_1B_2 are parallel to one another.

Thus by Theorem 50 $A_1'A_2'$ and A_1A_2 are parallel to one another.

THEOREM 104.

If a separation line a be conjugate to two intersecting inertia lines b and c , then any inertia line in the acceleration plane containing b and c is conjugate to a set of separation lines which are parallel to a .

Let the inertia lines b and c intersect in the element A_1 .

Then we know that a must also pass through A_1 , but does not lie in the acceleration plane containing b and c .

Let P_1 be the acceleration plane containing b and c ; let A_2 be any element in a distinct from A_1 and let P_2 be an acceleration plane through A_2 parallel to P_1 .

Then we have seen in the course of proving Theorem 102 that A_1 and A_2 are representatives of one another in the parallel acceleration planes P_1 and P_2 respectively, and further every inertia line in P_1 which passes through A_1 is conjugate to a .

Let d be any inertia line in the acceleration plane P_1 and let A_1' be any element in d while A_2' is the representative of A_1' in P_2 .

Then by Theorem 101 the separation line $A_1'A_2'$ is conjugate to d .

But, provided A_1' be distinct from A_1 , it follows by Theorem 103 that $A_1'A_2'$ is parallel to A_1A_2 : that is to a , and, since there are an infinite number of elements in d , it follows that d is conjugate to a set of separation lines which are parallel to a .

Thus the theorem is proved.

THEOREM 105.

If b and c be any two intersecting inertia lines, there is at least one separation line which is conjugate to both b and c .

Let the inertia lines b and c intersect in the element A_1 and let P_1 be the acceleration plane containing b and c .

Let any element be taken which does not lie in P_1 and through it let an acceleration plane P_2 be taken parallel to P_1 .

Let A_2 be the element in P_2 which is the representative of A_1 .

Then by Theorem 101 the separation line A_1A_2 is conjugate to any inertia line in P_1 which passes through A_1 .

Thus the separation line A_1A_2 is conjugate to both b and c , and so the theorem is proved.

THEOREM 106.

If b and c be any two intersecting separation lines such that no element of the one is either before or after any element of the other, there is at least one inertia line which is conjugate to both b and c .

Let the separation lines b and c intersect in the element A_1 and let Q be any acceleration plane containing b .

Now since no element of c is either *before* or *after* any element of b , it follows that c and b do not lie in one acceleration plane and therefore c does not lie in Q .

Let d_1 be the inertia line through A_1 in the acceleration plane Q which is conjugate to b .

Then d_1 being an inertia line which intersects c , it follows that d_1 and c lie in an acceleration plane, say R , which must be distinct from Q .

Let e be any other inertia line in R distinct from d_1 and passing through the element A_1 .

Then e being an inertia line which intersects b , it follows that e and b lie in an acceleration plane, say Q' , which must also be distinct from R .

Let d_1' be the inertia line through A_1 in the acceleration plane Q' which is conjugate to b .

Then d_1' may either coincide with e or be distinct from it.

Consider first the case where d_1' coincides with e .

Since then both d_1 and d_1' will lie in the acceleration plane R and since the separation line b is conjugate to both d_1 and d_1' , it follows, by Theorem 102, that b is conjugate to every inertia line in the acceleration plane R which passes through the element A_1 .

Let a be the inertia line through A_1 in the acceleration plane R which is conjugate to c .

Then a must also be conjugate to b and so the theorem will hold in this case.

Consider next the case where d_1' is distinct from e .

Since d_1 and d_1' are intersecting inertia lines, they will lie in an acceleration plane, say P_1 , which will be distinct from both Q and Q' .

But A_2 being an element of b is an element of the acceleration plane Q , and accordingly Q has the element A_2 in common with the acceleration plane P_2 .

Since however Q has the inertia line d_1 in common with P_1 , it follows by Theorem 46 that Q and P_2 have a general line, say d_2 , in common which must be parallel to d_1 and is therefore an inertia line.

But now since d_1 and d_2 are parallel inertia lines in the acceleration plane Q and since d_1 is conjugate to b , it follows that d_2 is conjugate to b .

Also since d'_1 and d'_2 are parallel inertia lines in the acceleration plane Q' and since d'_1 is conjugate to b , it follows that d'_2 is conjugate to b .

Thus the separation line b is conjugate to the two intersecting inertia lines d_2 and d'_2 in the acceleration plane P_2 and so by Theorem 102 b is conjugate to every inertia line in P_2 which passes through the element A_2 .

Now since no element of c is either *before* or *after* any element of b it follows that the element A_0 is neither *before* nor *after* the element A_2 and therefore, since A_2 and A_0 are distinct, A_2A_0 is a separation line.

Now let a_2 be the inertia line in the acceleration plane P_2 which passes through A_2 and is conjugate to A_2A_0 .

Then a_2 is also conjugate to b .

Thus if a_1 be an inertia line through A_1 parallel to a_2 it follows by Theorem 100 that a_1 is conjugate to A_1A_0 : that is to c .

But a_1 and a_2 being parallel inertia lines through elements of the separation line b and a_2 being conjugate to b , it follows that a_1 is also conjugate to b .

Thus a_1 is conjugate to both b and c and so the theorem is proved.

THEOREM 107.

If an inertia line a be conjugate to a separation line b , and if an inertia line a' be parallel to a while a separation line b' is parallel to b , and if a' and b' intersect one another, then a' is conjugate to b' .

Let P be the acceleration plane containing a and b and let A be the element of intersection of a and b , while A' is the element of intersection of a' and b' .

Two cases have to be considered:

- (1) A' lies in the acceleration plane P .
- (2) A' does not lie in the acceleration plane P .

Consider first case (1).

Here both a' and b' must lie in P and the element A' cannot lie either in a or b , since a' is parallel to a and b' is parallel to b .

Then since a' is parallel to a and a is conjugate to b , and since a , b and a' lie in one acceleration plane, it follows that a' is conjugate to b .

Also since b' is parallel to b and a' is conjugate to b , and since a' , b and b' lie in one acceleration plane, it follows that a' is conjugate to b' .

Consider next case (2).

Here a' and b' lie in an acceleration plane say P' which must be distinct from P , since the element A' does not lie in P .

Further, by Theorem 52, P' must be parallel to P .

Now since P and P' are parallel acceleration planes and since b is a general line in P , we know that there is at least one acceleration plane, say Q , containing b and another general line, say b_1 , in P' .

Then b_1 must be parallel to b and, since b is a separation line, b_1 must also be a separation line.

Further since a' is an inertia line and lies in the same acceleration plane P' with b_1 , it follows that a' and b_1 must intersect in some element, say A_1 .

Let c be an inertia line through A_1 in the acceleration plane Q and conjugate to b_1 and let c intersect b in A_2 .

Then the two intersecting inertia lines a' and c lie in an acceleration plane, say R , and, since R has the inertia line a' in common with P' and has the element A_2 in common with the parallel acceleration plane P , it follows, by Theorem 46, that R has a general line, say a_0 , in common with P , while a_0 must be parallel to a' and is therefore an inertia line.

But since a is parallel to a' , it follows that a_0 is either parallel to a or identical with it.

In either case, since a is conjugate to b , we must have a_0 conjugate to b .

But since c is conjugate to b_1 and since b is parallel to b_1 and lies in the acceleration plane Q along with c and b_1 , it follows that c is also conjugate to b .

Thus we have the separation line b conjugate to the two intersecting inertia lines a_0 and c and so by Theorem 104 any inertia line in the acceleration plane R containing a_0 and c is conjugate to a set of separation lines which are parallel to b .

But the inertia line a' lies in the acceleration plane R and therefore a' is conjugate to any separation line which intersects it and is parallel to b .

But b' intersects a' and is parallel to b and therefore a' is conjugate to b' .

Thus the theorem holds in all cases.

THEOREM 108.

If a be a separation line and B be any element which is not an element of a and is neither before nor after any element of a while c is a general line passing through B and parallel to a , then if A be any element of a , while C is an element of c distinct from B , a general line through C parallel to BA will intersect a .

Let the general line BA be denoted by b and let the general line through C parallel to b be denoted by d .

Then, as was pointed out in the remarks at the end of Theorem 99, no element of b is either *before* or *after* any element of a and so, since a and b intersect in A , it follows by Theorem 106 that there is at least one inertia line which is conjugate to both a and b , and must therefore pass through A .

Let a_1 be such an inertia line and let b_1 be an inertia line through B parallel to a_1 , while c_1 is an inertia line through C parallel to a_1 and b_1 .

Then a and a_1 lie in an acceleration plane which we may call P_a , while a_1 , b and b_1 lie in an acceleration plane which we may call P_b , and b_1 , c and c_1 lie in an acceleration plane which we may call P_c .

Then, since B and a do not lie in one acceleration plane, it follows that B is not an element of P_a and so, since b_1 and c are respectively parallel to a_1 and a , it follows that P_c is parallel to P_a .

Now since b_1 and a_1 both lie in P_b and are parallel to one another and since a_1 is conjugate to b , it follows that b_1 is also conjugate to b .

But since c is parallel to a while b_1 is parallel to a_1 , while c and b_1 intersect in B , it follows, by Theorem 107, that since a_1 is conjugate to a , therefore b_1 is conjugate to c .

Thus b_1 is conjugate to both b and c which are two distinct and intersecting separation lines and therefore cannot lie in one acceleration plane.

Thus C is not an element of P_b and so, if P_d be an acceleration plane containing c_1 and d , then, since c_1 is parallel to b_1 while d is parallel to b , it follows that the acceleration plane P_d is parallel to P_b .

We have next to show that the acceleration planes P_a and P_d have a general line in common.

Let e_1 be any inertia line in P_c which is parallel to b_1 and c_1 and therefore distinct from both.

Then e_1 will also be parallel to a_1 and so a_1 and e_1 will lie in an acceleration plane, say S .

Thus the three acceleration planes P_c , P_b and S and the three parallel general lines e_1 , b_1 and a_1 are such that e_1 lies in P_c and S , b_1 in P_b and P_c and a_1 in S and P_b , while the acceleration plane P_d is parallel to P_b and passes through the element C of P_c (which does not lie in b_1) and so, by Theorem 53, the acceleration planes S and P_d have a general line in common which is parallel to a_1 .

If we call this general line f_1 , then f_1 will be distinct from e_1 , for if f_1 were identical with e_1 we should have also c_1 identical with e_1 , contrary to hypothesis, while, since P_c and S are distinct acceleration planes, f_1 being distinct from e_1 must also be distinct from c_1 .

Thus f_1 is parallel to both e_1 and c_1 .

But now the three acceleration planes S , P_c and P_d and the three parallel general lines f_1 , e_1 and c_1 are such that f_1 lies in S and P_d , e_1 in P_c and S , and c_1 in P_d and P_c , while P_d is an acceleration plane parallel to P_c passing through an element A of S which does not lie in e_1 and so, by Theorem 53, the acceleration planes P_d and P_c have a general line in common which is parallel to c_1 .

If we call this general line d_1 , then, since P_d and P_b are parallel, d_1 must be parallel to a_1 and must be an inertia line.

Now since c_1 is parallel to b_1 and d is parallel to b and c_1 and d intersect, it follows, by Theorem 107, that, since b_1 is conjugate to b , therefore c_1 is conjugate to d .

Again since b_1 is conjugate to both b and c and since A is an element in b while C is a distinct element in c , it follows, by Theorem 100, that the general line c_1 is conjugate to CA .

Thus c_1 is conjugate to both d and CA .

Now since d is a separation line while d_1 is an inertia line and both lie in the acceleration plane P_d , it follows that d and d_1 must intersect in some element, say D .

Thus, since A is an element in CA while D is a distinct element in d , it follows, by Theorem 100, that a_1 is conjugate to DA .

But a_1 is conjugate to the separation line a which also passes through A , and so, since both DA and a lie in the acceleration plane P_a which contains a_1 , it follows that the general lines DA and a are identical.

Thus D lies in a and therefore the general line d intersects a .

Thus the theorem is proved.

REMARKS.

If a be a separation line and B be any element which is not an element of a and is neither *before* nor *after* any element of a , then if b

be a separation line through B parallel to a , no element of b is either *before* or *after* any element of a .

This is easily seen: for if C were an element of b which was either *before* or *after* an element of a , then the separation line a and the element C would lie either in one acceleration plane, or in one optical plane.

Such acceleration or optical plane would contain the general line through C parallel to a : that is to say it would contain b .

Thus the separation line a and the element B would lie in one acceleration or optical plane which we already know is impossible.

Thus no element of b is either *before* or *after* any element of a and therefore any general line which intersects both a and b must be a separation line.

Again, if AB and DC be two parallel separation lines such that no element of the one is either *before* or *after* any element of the other and if CB is parallel to DA , then no element of DA is either *before* or *after* any element of CB .

This is easily seen: for we know that no element of CB is either *before* or *after* any element of AB and therefore the element A is neither *after* nor *before* any element of CB .

Thus since DA is parallel to CB it follows that no element of DA is either *before* or *after* any element of CB .

THEOREM 109.

If A and B be two elements lying respectively in two parallel separation lines a and b which are such that no element of the one is either before or after any element of the other, and if A' be a second and distinct element in a , there is only one general line through A' and intersecting b which does not intersect the general line AB .

We have seen by Theorem 108 that the general line through A' parallel to AB must intersect b .

Let B' be the element of intersection. Then the general lines AB and $A'B'$, being parallel, cannot intersect.

Let any other general line through A' and intersecting b intersect it in the element C .

Then if C should coincide with B the general lines $A'C$ and AB have the element B in common and therefore intersect.

Suppose next that C does not coincide with B .

Since B is neither *before* nor *after* any element of a and since therefore no element of AB is either *before* or *after* any element of a , it

follows, by Theorem 106, that there is at least one inertia line, say a_1 , which is conjugate to both AB and a and therefore passes through A .

Let b_1 be an inertia line through B parallel to a_1 , and let a_1' and b_1' be inertia lines through A' and B' respectively and also parallel to a_1 .

Then a_1 and a_1' lie in one acceleration plane, say P_1 , which contains also the separation line a ; while b_1 and b_1' lie in an acceleration plane, say P_2 , containing b .

Since the elements B , A and A' cannot lie in one acceleration plane and since b_1 and b are respectively parallel to a_1 and a , it follows that P_2 is parallel to P_1 .

Again a_1 and b_1 lie in an acceleration plane, say Q , containing AB , while a_1' and b_1' lie in an acceleration plane, say Q' , containing $A'B'$.

Since the elements B , A and A' cannot lie in one acceleration plane and since a_1' and $A'B'$ are respectively parallel to a_1 and AB , it follows that Q' is parallel to Q .

Now the inertia line a_1' and the element C lie in an acceleration plane, say R , and so R has the element C in common with P_2 .

Thus we have the two parallel general lines a_1' and b_1' in the acceleration plane Q' and two other distinct acceleration planes R and P_2 containing a_1' and b_1' respectively and having the element C in common and therefore, by Theorem 51, R and P_2 have a general line in common, say c_1 , which is parallel to a_1' and b_1' .

But now Q is an acceleration plane through B , which is an element of P_2 not lying in b_1' , and Q is parallel to Q' and therefore, by Theorem 53, the acceleration planes R and Q have a general line in common, say f_1 , which is parallel to a_1' .

Now f_1 must be an inertia line and therefore will intersect the separation line AB in some element, say F , which must be distinct from A , since otherwise R would coincide with P_1 and could therefore have no element in common with P_2 , contrary to hypothesis.

Now since a_1 and b_1 both lie in Q and since a_1 is conjugate to AB , therefore b_1 is conjugate to AB .

Similarly a_1' is conjugate to $A'A$ or a .

Again, since b and b_1 intersect and since b is parallel to a while b_1 is parallel to a_1 , we must have b_1 conjugate to b .

Similarly a_1' is conjugate to $A'B'$.

Further, since b_1' lies in P_2 and is parallel to b_1 , therefore b_1' is conjugate to b ; and since b_1' lies in Q' and is parallel to a_1' , therefore b_1' is conjugate to $A'B'$.

But now since a_1 is conjugate to the two intersecting separation lines AB and a , and since F is an element in AB , while A' is a distinct

element in a , it follows by Theorem 100 that a_1' must be conjugate to $A'F$.

Again since b_1' is conjugate to the two intersecting separation lines $A'B'$ and b , and since A' is an element in $A'B'$ while C is a distinct element in b , it follows in a similar manner that a_1' must be conjugate to $A'C$.

Thus a_1' is conjugate to $A'F$ and to $A'C$ and since $A'F$ and $A'C$ each lie in the acceleration plane R and have an element in common, it follows that they must be identical.

Thus F lies in $A'C$ and also in AB and so $A'C$ intersects AB .

Thus there is only one general line through A' and intersecting b which does not intersect the general line AB .

THEOREM 110.

If a and b be two parallel separation lines such that no element of the one is either before or after any element of the other, and if one general line intersect a in A and b in B , while a second general line intersects a in A' and b in B' , then a general line through any element of AB and parallel to a or b intersects $A'B'$.

Let D be any element of AB and let d be a general line through D parallel to a or b .

We have to show that d intersects $A'B'$.

If D should coincide with A or B no proof is required and so we shall suppose it distinct from both.

If $A'B'$ be parallel to AB then no element of AB is either *before* or *after* any element of $A'B'$ and the result follows directly by Theorem 108.

If $A'B'$ be not parallel to AB then by Theorem 109 the general lines AB and $A'B'$ must intersect in some element, say C .

Now the general lines AB and $A'B'$ being supposed distinct, C must be distinct from at least one of the elements A and B and, without limitation of generality, we may suppose that C is distinct from B .

We shall then have B' distinct from B and so B' will not be an element of AB .

Thus through B' there is a parallel to AB and by Theorem 108 this parallel must intersect d in some element, say D' .

But now $D'B'$ and DB are parallel separation lines such that no element of the one is either *before* or *after* any element of the other and both are intersected by the general lines $D'D$ and $B'C$.

Further since we have supposed D to be distinct from B therefore D' is distinct from B' and so by Theorem 109 there is only one general line through B' and intersecting DB which does not intersect $D'D$.

But $B'B$ being parallel to $D'D$ must be this one general line and so since $B'C$ (that is $A'B'$) is distinct from $B'B$, it follows that $A'B'$ intersects $D'D$.

Thus in all cases a general line through any element of AB and parallel to a or b intersects $A'B'$.

THEOREM 111.

If a and b be two parallel separation lines such that no element of the one is either before or after any element of the other and if E be any element in a separation line AB which intersects a in A and b in B and if $A'B'$ be any other separation line intersecting a in A' and b in B' but not parallel to AB , then E either lies in $A'B'$ or in a separation line parallel to $A'B'$ which intersects both a and b .

If E does not lie in $A'B'$ then by Theorem 110 a separation line through E parallel to a or b intersects $A'B'$ in an element which is neither *before* nor *after* any element of a or b and so, by Theorem 108, a general line through E parallel to $A'B'$ intersects a and also b .

Thus E must lie in a separation line parallel to $A'B'$ and intersecting both a and b when it does not lie in $A'B'$ itself.

REMARKS.

If a and b be two parallel separation lines such that no element of the one is either *before* or *after* any element of the other and if c and d be any two non-parallel separation lines intersecting both a and b , then it is evident from Theorem 111 that the aggregate consisting of all the elements in c and in all separation lines intersecting a and b which are parallel to c must be identical with the aggregate consisting of all the elements in d and in all separation lines intersecting a and b which are parallel to d .

This follows since each element in the one set of separation lines must also lie in the other set.

Thus the aggregate which we obtain in this way is independent of the particular set of separation lines intersecting a and b which we may select and so we have the following definition.

Definition. If a and b be two parallel separation lines such that no element of the one is either *before* or *after* any element of the other, then the aggregate of all elements of all mutually parallel separation lines which intersect both a and b will be called a *separation plane**.

If a separation plane P be determined by the two parallel separation

* The name "separation plane" has been adopted from its analogy to a separation line.

lines a and b then any element C in P must lie in a separation line, say c , which intersects both a and b .

Any other element D in P must either lie in c or in a separation line, say d , parallel to c and intersecting both a and b .

If D lies in c then D is neither *before* nor *after* C .

If D lies in the separation line d we know that no element of d is either *before* or *after* any element of c and so again D is neither *before* nor *after* C .

Thus we have the general result that: *no element of a separation plane is either before or after any other element of it.*

THEOREM 112.

If two distinct elements of a general line lie in a separation plane then every element of the general line lies in the separation plane.

Let the separation plane be determined by the two parallel separation lines a and b which are such that no element of the one is either *before* or *after* any element of the other.

If the two given elements lie in a separation line which is known to intersect both a and b no proof is required.

Otherwise let C be any element in any separation line AB which intersects a in A and b in B and let D' be any element in any separation line $A'B'$ parallel to AB and intersecting a in A' and b in B' .

We have to show that every element of the general line CD' lies in the separation plane.

Now no element of AB is either *before* or *after* any element of $A'B'$ and so by Theorem 108 a general line through C parallel to a or b will intersect $A'B'$ in some element, say C' .

If D' should coincide with C' then CD' would be parallel to a or b and since C cannot be either *before* or *after* any element of a or b , it follows that no element of CD' could be either *before* or *after* any element of a or b .

Thus in this case, by Theorem 108, a general line through any element of CD' distinct from C taken parallel to AB will intersect both a and b .

Thus every element of CD' will in this case lie in the separation plane.

If D' should not coincide with C' then since CD' is distinct from CC' and intersects $A'B'$ it follows by Theorem 109 that CD' must intersect both a and b .

Thus again every element of CD' lies in the separation plane determined by a and b .

THEOREM 113.

If e be a general line in a separation plane and if A be any element of the separation plane which does not lie in e , then there is one single general line through A in the separation plane which does not intersect e .

We saw in the course of proving Theorem 112 that if a separation plane be determined by two parallel separation lines a and b such that no element of the one is either *before* or *after* any element of the other, then any general line containing two elements in the separation plane and therefore any general line lying in the separation plane must either be parallel to a or b , or else must intersect both a and b .

Suppose first that e intersects both a and b .

Since A does not lie in e it must lie in a general line d parallel to e and intersecting both a and b .

Now through A there is a separation line, say c , parallel to a or b and which, by Theorem 108, must intersect e and must lie in the separation plane, while any other general line f through A and lying in the separation plane must intersect both a and b .

Thus, by Theorem 109, f being supposed distinct from d must intersect e .

Suppose next that e is parallel to a or b .

Through A there is a separation line parallel to a or b and therefore parallel to e and which, as we know, lies in the separation plane.

Any other general line through A in the separation plane must intersect both a and b and so, by Theorem 110, it must be intersected by e .

Thus there is in all cases one single general line through A in the separation plane which does not intersect e .

THEOREM 114.

If A , B and C be three elements in a separation plane which do not all lie in one general line and if D be an element linearly between A and B , while E is an element linearly between B and C , there exists an element which lies both linearly between A and E and linearly between C and D .

The proof of this theorem is exactly analogous to that of Theorem 89 except that in this case the elements F and \bar{F} lie in the separation plane through A , B and C , so that, if they were distinct, F and \bar{F} would have to lie in a separation line.

But F and \bar{F} each lie in the inertia line f_1 and so it follows that they cannot be distinct.

Thus F is both linearly between A and E and linearly between C and D .

THEOREM 115.

If A , B and C be three elements in a separation plane which do not all lie in one general line and if D be an element linearly between A and B while F is an element linearly between C and D , there exists an element, say E , which is linearly between B and C and such that F is linearly between A and E .

The proof of this theorem is exactly analogous to that of Theorem 90 except that in this case the elements E and \bar{E} lie in the separation plane through A , B and C , so that, if they were distinct, E and \bar{E} would have to lie in a separation line.

But E and \bar{E} each lie in the inertia line e_1 and so it follows that they cannot be distinct.

Thus the element E is linearly between B and C and the element F is linearly between A and E .

REMARKS.

It will be observed that Theorem 114 is the analogue of Peano's axiom (14) for the case of elements in a separation plane, while Theorem 115 is the corresponding analogue of his axiom (13).

Further Theorem 113 corresponds to the Euclidean axiom of parallels for the case of general lines in a separation plane.

THEOREM 116.

If A , B and C be three elements in a separation plane which do not all lie in one general line and if D be the mean of A and B then :

(1) *A general line through D parallel to BC intersects AC in an element which is the mean of A and C .*

(2) *If E be the mean of A and C the general line DE is parallel to BC .*

The proof of this theorem is exactly analogous to that of Theorem 92.

It is to be noted however that for the case of a separation plane we can never have B' coinciding with B or C' coinciding with C since a separation plane cannot contain an optical line.

Definition. If a pair of parallel general lines in a separation plane be intersected by another pair of parallel general lines then the four general lines will be said to form a *general parallelogram in the separation plane*.

The terms *corner*, *side line*, *diagonal line*, *adjacent* and *opposite* will be used in a similar sense for the case of a general parallelogram in a separation plane as for one in an acceleration or optical plane.

THEOREM 117.

If we have a general parallelogram in a separation plane then :

(1) *The two diagonal lines intersect in an element which is the mean of either pair of opposite corners.*

(2) *A general line through the element of intersection of the diagonal lines and parallel to either pair of opposite side lines intersects either of the other side lines in an element which is the mean of the pair of corners through which that side line passes.*

The proof of this theorem is exactly analogous to that of Theorem 93 using Theorem 116 in place of Theorem 92.

THEOREM 118.

If A, B and C be three elements in a separation plane which do not all lie in one general line and if D be an element linearly between A and B while DE is a general line through D parallel to BC and intersecting AC in the element E, then E is linearly between A and C.

The proof of this theorem is exactly analogous to that of Theorem 78, using Theorem 114 in place of Theorem 76, and Theorem 115 in place of Theorem 77.

THEOREM 119.

If three parallel general lines a, b and c in one separation plane intersect a general line d_1 in A_1 , B_1 and C_1 respectively and intersect a second general line d_2 in A_2 , B_2 and C_2 respectively, then if B_1 is linearly between A_1 and C_1 we shall also have B_2 linearly between A_2 and C_2 .

The proof of this theorem is exactly analogous to that of Theorem 79, using Theorem 118 in place of Theorem 78.

THEOREM 120.

If three parallel general lines a, b and c in one separation plane intersect a general line d_1 in A_1 , B_1 and C_1 respectively and intersect a second general line d_2 in A_2 , B_2 and C_2 respectively, and if B_1 be the mean of A_1 and C_1 , then B_2 will be the mean of A_2 and C_2 .

The proof of this theorem is exactly analogous to that of Theorem 82, using Theorem 116 in place of Theorem 81, and Theorem 117 in place of Theorem 80.

THEOREM 121.

If A, B, C, D be the corners of a general parallelogram in a separation plane ; AB and DC being one pair of parallel side lines and BC and AD the other pair of parallel side lines, then if E be the mean of A and

B, while *F* is the mean of *D* and *C*, the general lines *AF* and *EC* are parallel to one another.

The proof of this theorem is exactly analogous to that of Theorem 98, using Theorem 117 in place of Theorem 80 or 93, and Theorem 116 in place of Theorem 81 or 92.

SETS OF THREE ELEMENTS WHICH DETERMINE SEPARATION PLANES.

If A_1 , A_2 and A_3 be three distinct elements which do not all lie in one general line and do not all lie in one acceleration plane or in one optical plane, then they must all lie in one separation plane, as we shall shortly show.

In those cases in which they do all lie in one separation plane they determine the separation plane containing them.

We have the following criterion by which we may say that the three elements do lie in one separation plane.

Three elements A_1 , A_2 , A_3 lie in one separation plane if A_1 and A_2 lie in a separation line while A_3 is an element which is not an element of the separation line and is neither *before* nor *after* any element of the separation line.

This is clearly true since if a be the separation line containing A_1 and A_2 , there is a separation line b through A_3 and parallel to a which is such that no element of b is either *before* or *after* any element of a .

The separation lines a and b then determine a separation plane which will contain A_1 , A_2 and A_3 .

If P be this separation plane it is the only one which contains A_1 , A_2 and A_3 , for suppose A_1 , A_2 and A_3 also lie in a separation plane P' determined by the two parallel separation lines a' and b' , which are such that no element of b' is either *before* or *after* any element of a' .

Now since P' contains A_1 , A_2 and A_3 it must contain the three general lines A_1A_2 , A_2A_3 and A_3A_1 , by Theorem 112.

At most only one of these general lines could be parallel to a' or b' .

Suppose first that A_1A_2 or a is not parallel to a' or b' .

Then a must intersect both a' and b' , and since A_3 is an element of P' the separation line b through A_3 parallel to a must lie in P' and must intersect both a' and b' .

Then every element in P must lie in a separation line intersecting both a and b and parallel to a' or b' .

But we know that every element of any such separation line c must

lie in P' , for by Theorem 110, a general line through any element of c parallel to a or b must intersect a' and b' .

Similarly every element in P' must lie in P and so P' must be identical with P .

Next suppose that a is parallel to a' or b' .

Then A_1A_3 cannot be parallel to a' or b' and so must intersect both a' and b' .

Then any element in P must lie either in A_3A_1 or in a general line parallel to A_3A_1 and intersecting both a and b .

But any such general line must also intersect both a' and b' and so every element in P must also lie in P' , and similarly every element in P' must also lie in P .

Thus again P' must be identical with P .

Thus there is only one separation plane containing the three elements A_1 , A_2 and A_3 .

Any three distinct elements A_1 , A_2 and A_3 which do not all lie in one general line must all lie either in an acceleration plane, an optical plane, or a separation plane.

This is easily seen; for A_1 and A_2 must lie either in an optical line, an inertia line, or a separation line.

If A_1A_2 be an optical line we must have either

(1) A_3 *after* an element of A_1A_2 ,

or (2) A_3 *before* an element of A_1A_2 ,

or (3) A_3 neither *before* nor *after* any element of A_1A_2 .

We cannot have A_3 *after* one element of A_1A_2 and *before* another element of it, since A_3 is not an element of A_1A_2 (Theorem 12).

In cases (1) and (2), as we have seen, A_1 , A_2 and A_3 lie in an acceleration plane.

In case (3) we have seen that A_1 , A_2 and A_3 lie in an optical plane.

If A_1A_2 be an inertia line we know that the three elements must always lie in an acceleration plane.

If A_1A_2 be a separation line we must have either

(1) A_3 *after* at least two distinct elements of A_1A_2 ,

or (2) A_3 *before* at least two distinct elements of A_1A_2 ,

or (3) A_3 *after* one single element of A_1A_2 ,

or (4) A_3 *before* one single element of A_1A_2 ,

or (5) A_3 neither *before* nor *after* any element of A_1A_2 .

We cannot have A_3 *after* one element of A_1A_2 and *before* another element of it for then we should have one element of A_1A_2 *after* another element of it, contrary to the hypothesis that A_1A_2 is a separation line.

We have already seen that in cases (1) and (2) A_1 , A_2 and A_3 lie in an acceleration plane.

Also in cases (3) and (4) we have seen that A_1 , A_2 and A_3 lie in an optical plane.

Finally in case (5) we have seen that A_1 , A_2 and A_3 lie in a separation plane.

This exhausts all the possibilities which are logically open and so we see that A_1 , A_2 and A_3 must always lie either in an acceleration plane, an optical plane, or a separation plane.

It follows directly that any two intersecting general lines a and b must lie either in an acceleration plane, an optical plane, or a separation plane, which we may call P .

Any element in P must lie either in b or in a general line parallel to b and intersecting a .

Also conversely, any element in b or in any general line which intersects a and is parallel to b , must lie in P .

Thus we have the following definition:

Definition. If a and b be any two intersecting general lines then the aggregate of all elements of the general line b and of all general lines parallel to b which intersect a will be called a *general plane*.

Thus a general plane is a common designation for an acceleration plane, an optical plane, or a separation plane.

By combining Theorems 76, 89 and 114 we now see that the analogue of Peano's axiom (14) holds in general for our geometry; while by combining Theorems 77, 90 and 115 we see that the analogue of his axiom (13) also holds in general.

Again by combining Theorems 47, 88 and 113 we get what corresponds to the Euclidean axiom of parallels for the case of general lines in a general plane.

Peano's fifteenth axiom is as follows:

A point can be found external to any plane.

It is evident in our geometry that, since there is more than one general plane, there is an element external to any general plane, and so the analogue of Peano's axiom (15) also holds.

If a and b be two intersecting general lines in a general plane P and if through any element A not lying in P two general lines a' and b' be taken respectively parallel to a and b , then if P' be the general plane

determined by a' and b' , the two general planes P and P' can have no element in common.

This is easily seen, for in the first place the general line a' can have no element in common with P , for then, since it is parallel to a , every element of a' would have to lie in P , contrary to the hypothesis that the element A does not lie in P .

Similarly b' can have no element in common with P .

If now B be any element in a' distinct from A and if b'' be a general line through B parallel to b' then b'' must also be parallel to b and since B does not lie in P it follows that b'' can have no element in common with P .

But any element in P' must lie either in b' or in a general line parallel to b' which intersects a' and therefore the general plane P' can have no element in common with P .

THEOREM 122.

If an inertia line a be conjugate to two intersecting separation lines b and c , then b and c lie in a separation plane such that any separation line in it is conjugate to a set of inertia lines which are parallel to a .

Let the separation lines b and c intersect in the element A .

Then we know that a must also pass through A and that the separation lines b and c must be such that no element of the one is either *before* or *after* any element of the other, and so there must be a separation plane, say P , which contains them.

Let B and C be elements in b and c respectively and let them both be distinct from A .

Then BC is a separation line which we may call d and which lies in the separation plane P .

Let e be an inertia line through B parallel to a .

Then e is conjugate to b and, by Theorem 100, it must also be conjugate to d .

Now we know that there is only one general line in P and passing through A which does not intersect d .

Let AF be any general line passing through A and intersecting d in F .

Then, by Theorem 100, since e is conjugate to b and d , it follows that a must be conjugate to AF .

Again if d' be the general line through A parallel to d , it must lie in the separation plane P , and since e is conjugate to d , while a and d' are respectively parallel to e and d , and since a and d' intersect one another, it follows by Theorem 107 that a must be conjugate to d' .

Thus every separation line passing through A in the separation plane P is conjugate to a and therefore also conjugate to any inertia line which intersects it and is parallel to a .

Consider now any separation line f in P which does not pass through A .

Then there is a separation line f' passing through A and parallel to f , and a must be conjugate to f' .

Thus by Theorem 107 any inertia line intersecting f and parallel to a must be conjugate to f .

Thus any separation line in P , whether it pass through A or not, must be conjugate to a set of inertia lines which are parallel to a and so the theorem is proved.

THEOREM 123.

If a and b be any two intersecting general lines in a general plane P and if through any element O' not lying in P two general lines a' and b' respectively parallel to a and b be taken determining a general plane P' , then there is a general line through O' and lying in P' which is parallel to any general line in P .

Let the general lines a and b intersect in the element O and let A and B be any two elements distinct from O and lying in a and b respectively.

Then the general lines OO' and a determine a general plane which must contain a' , since a' is parallel to a and intersects OO' .

Thus a general line through A parallel to OO' will intersect a' in some element, say A' .

Similarly a general line through B parallel to OO' will intersect b' in some element, say B' .

Then BB' will be parallel to AA' .

But AB and AA' determine a general plane which must contain BB' and so the general lines AB and $A'B'$ must lie in one general plane.

But AB lies in P while $A'B'$ lies in P' , and so $A'B'$ can have no element in common with AB and must therefore be parallel to it.

Let the general line AB be denoted by c and the general line $A'B'$ by c' .

Let c_1 be a general line through O parallel to c while c_1' is a general line through O' parallel to c' .

Then c_1 will lie in P and c_1' will lie in P' , and since c' is parallel to c we must also have c_1' parallel to c_1 .

Now any general line in P and passing through O , with the exception of c_1 , must intersect c in some element, say X .

If X should coincide with either A or B we know that $O'A'$ and $O'B'$ are respectively parallel to OA and OB , so that we shall suppose X distinct from A and B .

If now a general line be taken through X parallel to AA' such general line will lie in the general plane determined by AB and AA' and will therefore intersect $A'B'$ in some element, say X' .

Now XX' must be parallel to OO' and so XX' must lie in the general plane determined by OX and OO' .

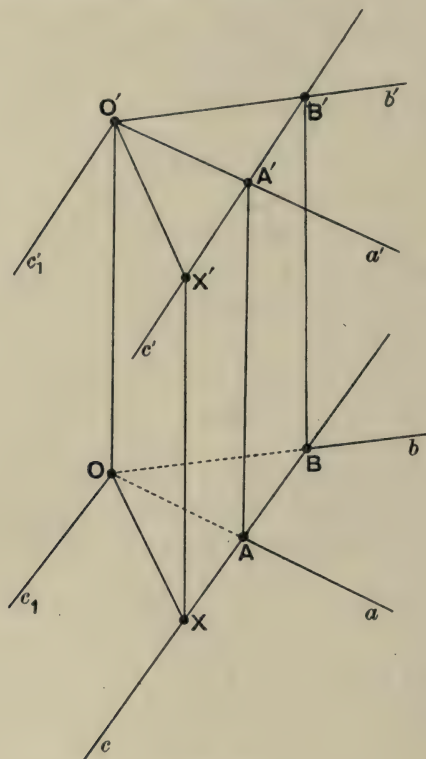


Fig. 33.

Thus OX and $O'X'$ lie in one general plane.

But OX lies in P while $O'X'$ must lie in P' and, since P and P' have no element in common, it follows that $O'X'$ is parallel to OX .

Thus through O' there is a general line in P' which is parallel to any general line in P which passes through O , and since any general line in P which does not pass through O is parallel to one which does pass through O , it follows that there is a general line through O' and lying in P' which is parallel to any general line in P .

It also follows directly from the above that through *any* element of P' there is a general line in P' which is parallel to any general line in P .

REMARKS.

We have already given a definition of the parallelism of acceleration planes and are now in a position to give a definition of the parallelism of general planes which will include that of acceleration planes as a special case.

Definition. If P be a general plane and if through any element A outside P two general lines be taken respectively parallel to two intersecting general lines in P , then the two general lines through A determine a general plane which will be said to be *parallel* to P .

Theorem 52 shows that this definition agrees with that given for the case of acceleration planes.

If P be a general plane and A be any element outside it, while P' is a general plane through A parallel to P , then it is evident from Theorem 123 that, since P' contains the general line through A parallel to any general line in P , the general plane P' must be uniquely determined when we know P and A .

Thus *through any element outside a general plane P there is one single general plane parallel to P .*

Also it is clear that this general plane must be of the same kind as P .

Again since two distinct general lines which are parallel to a third general line are parallel to one another, it follows that: *two distinct general planes which are parallel to a third general plane are parallel to one another.*

Definition. If P be a general plane and if through any element A outside P a general line a be taken parallel to any general line in P , then the general line a will be said to be *parallel* to the general plane P .

THEOREM 124.

If a general plane P have one element in common with each of a pair of parallel general planes Q and R then, if P have a second element in common with Q it also has a second element in common with R .

Let the general plane P have the element A in common with Q and the element A' in common with R .

Further let P and Q have a second element B in common.

Then, as was observed at the end of Theorem 123, there is a general line, say c , through A' in the general plane R which is parallel to AB .

But c must also lie in P , and so any element of c distinct from A' is a second element common to P and R .

Thus the theorem is proved.

THEOREM 125.

If two parallel general lines a and b lie in one general plane R and if two other distinct general planes P and Q containing a and b respectively have an element A_1 in common, then P and Q have a general line in common which is parallel to a and b .

The proof of this theorem is exactly analogous to that of Theorem 51, using Theorem 124 in place of Theorem 46.

THEOREM 126.

If three distinct general planes P , Q and R and three parallel general lines a , b and c be such that a lies in P and R , b in Q and P and c in R and Q , then if Q' be a general plane parallel to Q through some element of P which does not lie in b the general planes R and Q' have a general line in common which is parallel to c .

The proof of this theorem is analogous to that of Theorem 53, using Theorem 124 in place of Theorem 46.

Since however a general plane does not always contain an optical line, we take any general line through the element A distinct from a , which lies in the general plane P , and such general line must intersect b in an element which we shall call B .

We then take any general line through B distinct from b , which lies in the general plane Q , and this general line must intersect c in some element which we shall call C .

Then BA and BC lie in a general plane which we shall call S .

The demonstration from this point on is similar to that of Theorem 53, once more using Theorem 124 in place of Theorem 46.

If a pair of parallel general lines be both intersected by another pair of parallel general lines then the four general lines will form a *general parallelogram* either in an *acceleration plane*, an *optical plane*, or a *separation plane*.

Thus a *general parallelogram* may now be defined in this way without specifying which type of general plane it lies in.

THEOREM 127.

If two general parallelograms have a pair of adjacent corners A and B in common, their remaining corners either lie in one general line or else form the corners of another general parallelogram.

Let A, B, C, D be the corners of the one general parallelogram and A, B, C', D' the corners of the other, and let AC and BD be a pair of opposite side lines of the first general parallelogram while AC' and BD' are a pair of opposite side lines of the second.

Then CD and $C'D'$ being each parallel to AB must either be parallel to one another or else must be identical.

In the latter case the corners C, D, C', D' lie in one general line.

Suppose now that CD and $C'D'$ are distinct and therefore parallel; we have to prove that CC' is parallel to DD' .

Two cases have to be considered:

- (1) The two general parallelograms lie in distinct general planes,
- or (2) The two general parallelograms lie in the same general plane.

We shall first consider case (1).

Since CD and $C'D'$ are parallel they must lie in a general plane, say P .

Again AC and AC' must lie in a general plane, say Q , distinct from the general planes of either of the general parallelograms, since by hypothesis C' does not lie in the general plane containing A, B, C and D .

Similarly BD and BD' must lie in a general plane, say R , distinct from the general planes of either of the general parallelograms.

Further the element A cannot lie in R , since otherwise A, B, C, D, C' and D' would all lie in one general plane, contrary to hypothesis.

But AC is parallel to BD , while AC' is parallel to BD' and therefore Q is parallel to R .

Thus the general lines CC' and DD' can have no element in common, and since they both lie in P , it follows that they are parallel.

Thus C, C', D', D form the corners of another general parallelogram.

We have next to consider case (2).

Let P be the general plane containing the two given general parallelograms, and let Q be any other general plane distinct from P and containing the general line AB .

Let AC_1 be any general line distinct from AB which passes through A and lies in Q .

Through any element C_1 of AC_1 distinct from A let a general line be taken parallel to AB and let it meet the general line through B parallel to AC_1 in the element D_1 .

Then by the case already proved the general lines CC_1 and DD_1 are parallel.

Similarly $C'C_1$ and $D'D_1$ are parallel to one another.

But now the general parallelograms whose corners are C_1, D_1, D, C and C_1, D_1, D', C' cannot lie in one general plane; for the general lines CD and $C'D'$ both lie in P , while C_1D_1 does not lie in P .

Thus again by case (1) CC' is parallel to DD' and so C, C', D', D form the corners of a general parallelogram.

Thus the theorem holds in all cases.

THEOREM 128.

(1) *If three distinct elements A, B and C in a general plane P do not all lie in one general line and if D be any element linearly between B and C , then any general line passing through D and lying in P and which is distinct from BC and AD must either intersect AC in an element linearly between A and C , or else intersect AB in an element linearly between A and B .*

(2) *If further E be an element linearly between C and A and if F be an element linearly between A and B , then D, E and F cannot lie in one general line.*

In order to prove the first part of the theorem let a be any general line passing through D and lying in P .

Then a must either be parallel to AC or else intersect AC in some element, say E .

If a be parallel to AC then it follows by Theorems 78, 94 and 118 that a must intersect AB in an element which is linearly between A and B .

If a intersects AC in an element E then provided a be distinct from BC and AD we must either have:

- (i) E linearly between A and C ,
- or (ii) C linearly between A and E ,
- or (iii) A linearly between C and E .

In case (ii) it follows by the analogue of Peano's axiom (13) that a intersects AB in an element linearly between A and B , while in case (iii) it follows by the analogue of Peano's axiom (14) that a intersects AB in an element linearly between A and B .

Thus the first part of the theorem is proved.

In order now to prove the second part of the theorem it is to be observed in the first place that since the elements D, E and F lie in three distinct general lines BC, CA and AB and are distinct from the

elements of intersection of these, therefore the elements D , E and F are all distinct.

If then D , E and F lay in one general line, we should require to have either:

- E linearly between D and F ,
- or F linearly between E and D ,
- or D linearly between F and E .

Now the elements F , C and B do not lie in one general line and we have D linearly between B and C .

If then we had also E linearly between D and F it would follow that A must be linearly between B and F , contrary to the hypothesis that F is linearly between A and B .

Thus E cannot be linearly between D and F .

Similarly F cannot be linearly between E and D and further D cannot be linearly between F and E .

It follows therefore that D , E and F cannot lie in one general line and so the second part of the theorem is proved.

THEOREM 129.

(a) *If an element B be linearly between two elements A and C and if another element D be before both A and C but not in the general line AC , then DB is an inertia line and B is after D .*

Consider first the case where AC is a separation line.

Let a general line through B parallel to CD intersect AD in E .

Then since B is linearly between A and C we must have E linearly between D and A .

Thus since D is *before* A it follows that E is *after* D and *before* A .

But EB must be an inertia line or an optical line according as DC is an inertia line or an optical line and so B must be either *before* or *after* E .

But B cannot be *before* E for then, since E is *before* A , we should have B *before* A , contrary to the hypothesis that A and B lie in a separation line.

Thus B must be *after* E and, since E is *after* D , it follows that B is *after* D .

Thus DB is either an optical or an inertia line.

But if DB were an optical line then since E is *after* D and *before* B it would follow that E would lie in DB , which is impossible since BE is parallel to CD .

Thus DB must be an inertia line.

Next consider the case where AC is an optical or inertia line.

We then must have either C after A or A after C and it is sufficient to consider the case where C is after A .

Then B must be after A and before C .

But A is after D and so B must be after D .

Thus again DB must be either an optical line or an inertia line.

If DB were an optical line, then since A is after D and before B the element A would have to lie in DB and so D , A and C would all lie in one general line, contrary to hypothesis.

Thus again DB must be an inertia line, and so the theorem is proved.

(b) If an element B be linearly between two elements A and C and if another element D be after both A and C but not in the general line AC , then DB is an inertia line and B is before D .

THEOREM 130.

If an optical line a and an inertia line b intersect in an element A and if a separation line c passing through A be such that no element of c except A is either before or after any element of a and if further c be conjugate to b , then c is conjugate to every inertia line which passes through A and lies in the acceleration plane containing a and b .

Let A' be any element of c distinct from A and let a' be an optical line through A' parallel to a while b' is an inertia line through A' parallel to b .

Then a' must be a neutral-parallel of a .

Let P be the acceleration plane containing a and b and let P' be the acceleration plane containing a' and b' .

Then, since A' is neither before nor after any element of the optical line a , it follows that A' does not lie in P and so the acceleration planes P and P' are parallel to one another.

Let \bar{A}' be the representative of A in P' .

Then \bar{A}' must lie in a' by the definition of representative elements and by Theorem 101 $A\bar{A}'$ is conjugate to b .

But AA' is conjugate to b and so if \bar{A}' were distinct from A' we should have b conjugate to two intersecting separation lines and so by Theorem 99 no element of AA' could be either before or after any element of $A\bar{A}'$.

But, if \bar{A}' were distinct from A' , then, since they each lie in the optical line a' , it would follow that the one must be after the other.

Thus the supposition that \bar{A}' is distinct from A' leads to a contradiction and so \bar{A}' must be identical with A' .

Thus it follows by Theorem 101 that AA' (that is c) is conjugate to every inertia line which passes through A and lies in P .

Thus the theorem is proved.

THEOREM 131.

If O be any element in a separation line b lying in a separation plane P and if a be an inertia line through O which is conjugate to every separation line in P which passes through O , then there is one and only one such separation line which is conjugate to every inertia line passing through O and lying in the acceleration plane containing a and b .

Let Q be the acceleration plane containing a and b and let Q' be an acceleration plane parallel to Q through any element of P which does not lie in b .

Then by Theorem 124 P and Q' will have a general line, say b' , in common which must be parallel to b and must be a separation line.

Let c be one of the generators of Q which pass through O .

Then since Q' is parallel to Q there is one single generator of Q' , say c' , which is neutrally parallel to c .

Let c' intersect b' in O' .

Then O' is neither *before* nor *after* any element of c and so no element of the general line OO' with the exception of O is either *before* or *after* any element of c .

But OO' lies in P and therefore is conjugate to a and so by Theorem 130 OO' is conjugate to every inertia line in Q which passes through O .

Thus, as in Theorem 130, O and O' are representatives of one another in the parallel acceleration planes Q and Q' , and further, we may show as in Theorem 102 that if O'' be any element of Q' distinct from O' the general line OO'' cannot be conjugate to two distinct inertia lines in Q which pass through O .

But now any separation line in P which passes through O must either be identical with b or else intersect b' in some element.

If it should intersect b' in any element other than O' it cannot be conjugate to more than one inertia line in Q which passes through O .

Also if it be identical with b it cannot be conjugate to more than one inertia line in Q which passes through O .

Thus there is one and only one separation line in P which passes through O and is conjugate to every inertia line passing through O and lying in the acceleration plane Q .

corners of an optical parallelogram of which O is the centre, and since B_1 is *after* A_1 it must also be *after* O .

Let the separation line through B_1 parallel to b intersect the inertia line c in the element C .

Then B_1C is conjugate to c .

Let D be the one single element common to c and the α sub-set of B_1 and let E be the one single element common to c and the β sub-set of B_1 .

Then B_1D and B_1E are optical lines and so D , B_1 and E are three corners of an optical parallelogram, which has C as centre, since B_1C is conjugate to c .

Let B_2 be the remaining corner of this optical parallelogram.

Then B_2 lies in B_1C .

Let DB_2 intersect the optical line through O parallel to B_1D in F .

Now we have seen in the course of proving Theorem 102, that if we take an acceleration plane through A_1 parallel to P then A_1 and O are representatives of one another in these two parallel acceleration planes and so A_1 is neither *before* nor *after* any element of either of the generators of P which pass through O .

Since OF is one of these generators, it follows that A_1 is neither *before* nor *after* the element F .

But B_1 is *after* A_1 and D is *after* B_1 and so D is *after* A_1 .

Thus since A_1 is not an element of the optical line DF but is *before* an element of it, it follows that there is one single element common to the optical line DF and the α sub-set of A_1 .

Let B_2' be this element.

Then since D is *after* A_1 but does not lie in an optical line with it, it follows that B_2' cannot be either identical with D or *after* D .

Also since F is not *after* A_1 , it follows that B_2' is not either identical with F or *before* F .

Thus B_2' must be *after* F and *before* D .

Now D being *after* B_1 and B_1 *after* O we must have D *after* O , and since FO and FD are both optical lines we must have F *after* O .

Thus, since B_2' is *after* F , it follows that B_2' is *after* O ; and so since B_2' and O do not lie in one optical line, OB_2' must be an inertia line.

Since then OB_2' lies in P and passes through O , the separation line a must be conjugate to it.

Thus since O is the mean of A_1 and A_2 it follows that A_1 , B_2' and A_2 are three corners of an optical parallelogram of which O is the centre.

Let an optical line through B_2' parallel to DB_1 intersect the optical line B_1E in E' .

Then B_1, D, B_2', E' form the corners of an optical parallelogram.

Let the diagonal lines B_1B_2' and DE' intersect in C' ; then C' is the mean of B_1 and B_2' .

Thus since A_1B_1 and A_1B_2' are optical lines it follows that B_1, A_1 and B_2' are three corners of an optical parallelogram having C' as centre.

Similarly, since A_2B_1 and A_2B_2' are optical lines, B_1, A_2 and B_2' are three corners of an optical parallelogram having C' as centre.

But D is *after* B_1 and also *after* B_2' and so, since DB_1 and DB_2' are distinct optical lines, it follows that B_2' is neither *before* nor *after* B_1 and therefore B_1B_2' is a separation line.

Thus $C'A_1$ and $C'A_2$ are each inertia lines and each must be conjugate to B_1B_2' .

It follows, by Theorem 102, that B_1B_2' is conjugate to every inertia line in the acceleration plane containing $C'A_1$ and $C'A_2$ which passes through C' .

Then since the element O lies in A_1A_2 , it follows that $C'O$ must lie in this acceleration plane and must therefore be conjugate to B_1B_2' provided it be an inertia line.

But C' is the mean of B_1 and B_2' and is therefore linearly between B_1 and B_2' while the element O is *before* both B_1 and B_2' .

Thus, since O is not in the general line B_1B_2' , it follows by Theorem 129 (a) that $C'O$ is an inertia line.

Thus B_1B_2' is conjugate to $C'O$.

But B_1B_2' is conjugate to $C'D$ and so, since $C'O$ and $C'D$ each lie in the acceleration plane P , it follows that they are identical.

Thus C' lies in OD and therefore E' must be identical with E and accordingly B_2' must be identical with B_2 .

Further C' must be identical with C and so B_1B_2 is conjugate to both CA_1 and CA_2 and therefore also conjugate to every inertia line in the acceleration plane containing CA_1 and CA_2 which passes through C .

But this is the acceleration plane containing a and c , and the separation line b is parallel to B_1B_2 and passes through O and so it follows from Theorem 107 that any inertia line in this acceleration plane which passes through O must be conjugate to b .

Thus the theorem is proved.

REMARKS.

All the postulates which have hitherto been introduced may be represented by ordinary geometric figures involving not more than three dimensions.

This may be done in the manner described in the introduction: the α and β sub-sets being represented by cones.

We have now however to introduce a new postulate which cannot be represented along with the others in a three-dimensional figure and which therefore gives our geometry a sort of four-dimensional character.

The new postulate is as follows:

POSTULATE XIX. If P be any optical plane there is at least one element which is neither before nor after any element of P .

Since any element in an optical plane must lie in a generator, it will be *after* certain elements and *before* certain other elements of that optical plane.

It follows that any element such as is postulated in Post. XIX must lie outside P .

If P be an optical plane and A be any element which is neither *before* nor *after* any element of P , then an optical line through A parallel to any generator of P will be a neutral-parallel and accordingly any generator of an optical plane lies in at least one other distinct optical plane.

Since we already know that any optical line lies in at least one optical plane, it follows that *there are at least two distinct optical planes containing any optical line*.

This might be taken as an alternative form of the postulate.

If P and Q be two distinct optical planes having an optical line a in common, then any element of Q which does not lie in a must lie in a generator of Q , say b , which is a neutral-parallel of a .

Since any generator of P which is distinct from a is also a neutral-parallel of a , it follows by Theorem 27 that b is a neutral-parallel of every generator of P .

Since every element of P lies in a generator it follows that no element of Q lying outside a is either *before* or *after* any element of P .

Although Post. XIX is required in order to prove that there are at least two distinct optical planes containing any optical line, it is possible, without using this postulate, to prove that there are at least two distinct optical planes containing any separation line.

This may be done in the following manner:

Let b be the separation line and O be any element in it.

We already know that if we take any two acceleration planes containing b , then b is conjugate to one single inertia line in each of them which passes through O .

If a_1 and a_2 be two such inertia lines, then, as was shown in Theorem 102, b is conjugate to every inertia line in the acceleration plane containing a_1 and a_2 which passes through O .

Further if c_1 and c_2 be the two generators of this acceleration plane which pass through O it was also shown in the course of proving Theorem 102 that if we take any element O' of b distinct from O such element is neither *before* nor *after* any element of either c_1 or c_2 .

Thus if we take an optical line through O' parallel to c_1 it will be a neutral-parallel and so b and c_1 lie in an optical plane.

Similarly b and c_2 lie in an optical plane.

These optical planes must be distinct since c_1 and c_2 are distinct optical lines which both pass through O .

THEOREM 133.

If b be any separation line and O be any element in it, there are at least two acceleration planes containing O and such that b is conjugate to every inertia line in each of them which passes through O .

Let Q be an optical plane containing b and let c_1 be the generator of Q which passes through O .

Then by Post. XIX it follows, as we have already shown, that there is at least one other optical plane, say R , containing the optical line c_1 .

Let d be any separation line in R and passing through O .

Then no element of d except O is either *before* or *after* any element of Q and O itself is neither *before* nor *after* any element of Q which lies outside c_1 .

Thus no element of d is either *before* or *after* any element of b and so, by Theorem 106, there is at least one inertia line, say a , which is conjugate to both b and d .

Thus, as was shown in Theorem 122, a must be conjugate to every separation line which lies in the separation plane containing b and d and which passes through O .

Let P be the separation plane containing b and d .

Then by Theorem 131 there is one and only one separation line, say e , lying in P and passing through O which is conjugate to every inertia line passing through O and lying in the acceleration plane containing a and b .

Let T be the acceleration plane containing a and e .

Then, by Theorem 132, since b is conjugate to a , it follows that b is conjugate to every inertia line in T which passes through O .

Now let U be the acceleration plane containing a and c_1 .

Then since b is conjugate to a and since no element of b with the exception of O is either *before* or *after* any element of c_1 it follows, by Theorem 130, that b is conjugate to every inertia line in U which passes through O .

Similarly since d is conjugate to a and since no element of d with the exception of O is either *before* or *after* any element of c_1 , it follows that d is conjugate to every inertia line in U which passes through O .

Thus any inertia line in U which passes through O is conjugate to both b and d and therefore is conjugate to every separation line passing through O and lying in the separation plane P .

It follows that U cannot have more than one element in common with P , for if it had it would have a separation line in common with P and every inertia line in U which passed through O would be conjugate to one separation line lying in U , which is impossible.

But now b is conjugate to every inertia line lying either in T or U which passes through O and since T contains the separation line e which lies in P , while U does not contain any separation line in P , it follows that T and U are distinct acceleration planes.

Thus the theorem is proved.

REMARKS.

It follows from this that if a separation line b have an element O in common with any acceleration plane T and is conjugate to every inertia line in T which passes through O , then b is also conjugate to certain other inertia lines passing through O which do not lie in T .

It also follows directly that there are certain optical lines passing through O , but not lying in T , which are such that no element of b with the exception of O is either *before* or *after* any element of them.

Another important point which arises in the last theorem is that we may have an acceleration plane and a separation plane having only one element in common and such that each inertia line through the common element in the former is conjugate to every separation line through it in the latter.

THEOREM 134.

If two distinct acceleration planes P and P' have a separation line b in common and if another separation line c intersecting b in the element O be conjugate to every inertia line in P which passes through O , then if c be conjugate to one inertia line in P' which passes through O it is conjugate to every inertia line in P' which passes through O .

Let f_1 and f_2 be the two generators of P which pass through O and let D_1 be any element in f_1 which is *after* O .

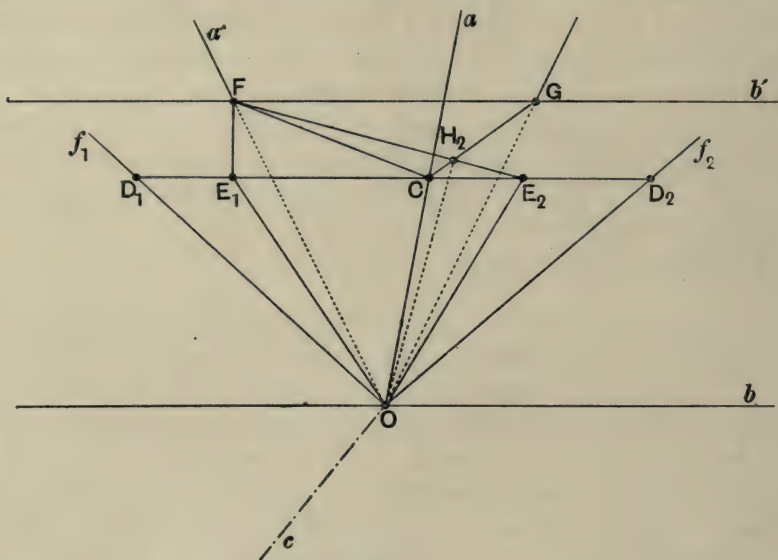
Let the general line through D_1 parallel to b intersect f_2 in D_2 .

Then D_1D_2 is a separation line and so since O is *before* D_1 it must also be *before* D_2 .

Let E_1 be any element linearly between D_1 and D_2 and let E_2 be any element linearly between E_1 and D_2 , while C is any element linearly between E_1 and E_2 .

Then by Theorem 129 (a) OE_1 is an inertia line and E_1 is *after* O .

Similarly, since O is *before* both E_1 and D_2 , it follows that OE_2 is an inertia line and E_2 is *after* O , and further since O is *before* both E_1 and E_2 it follows that OC must be an inertia line and C must be *after* O .



[Fig. 35.]

Thus OE_1 , OE_2 and OC are three distinct inertia lines in P all passing through O and so the separation line c is conjugate to each of them.

Now if a' be an inertia line in P' which passes through O and to which c is conjugate, it follows by Theorem 102 that c is conjugate to every inertia line passing through O and lying in either of the three acceleration planes containing a' and OE_1 , a' and OE_2 or a' and OC .

Let F be any element of a' which is *after* O and let b' be the general line through F parallel to b .

Then b' must lie in P' and must be parallel to D_1D_2 .

Let Q be the general plane containing b' and FC .

Then Q contains D_1D_2 and therefore also contains FE_1 and FE_2 .

Now any general line in P' which passes through O with the exception of b must intersect b' in some element, say G .

If now we consider the general line CG , we see that it must lie in Q since C and G are distinct elements in Q .

Further CG must be distinct from E_1E_2 since E_1E_2 is parallel to b' while CG intersects b' .

Thus since F, E_1 and E_2 do not lie in one general line and since C is linearly between E_1 and E_2 , it follows by Theorem 128 that, provided G does not coincide with F , the general line CG either intersects FE_2 in an element linearly between F and E_2 or else intersects FE_1 in an element linearly between F and E_1 .

Consider the case where CG intersects FE_2 in an element H_2 linearly between F and E_2 .

Then, since O is *before* both F and E_2 , it follows by Theorem 129 that OH_2 is an inertia line, and since it lies in the acceleration plane containing a' and OE_2 and passes through O , it follows that c is conjugate to it.

But c is also conjugate to OC and so by Theorem 102 c is conjugate to every inertia line in the acceleration plane containing OC and OH_2 which passes through O .

Similarly if CG should intersect FE_1 in an element H_1 linearly between F and E_1 , then c is conjugate to every inertia line in the acceleration plane containing OC and OH_1 which passes through O .

Thus in either case if OG should happen to be an inertia line, c must be conjugate to it.

Thus c must be conjugate to every inertia line in P' which passes through O and so the theorem is proved.

REMARKS.

Since, in the above theorem, there is one single inertia line through O in the acceleration plane P which is conjugate to b , such inertia line will be conjugate to both b and c and so it follows, by Theorem 99, that no element of b is either *before* or *after* any element of c and so b and c must lie in a separation plane.

Again if f_1' and f_2' be the two generators of P' which pass through O , then no element of c with the exception of O will be either *before* or *after* any element of either f_1' or f_2' .

Now let a_1 be the one single inertia line through O and lying in P which is conjugate to b , and let a_1' be the one single inertia line through O and lying in P' which is conjugate to b .

Then a_1 and a_1' lie in an acceleration plane, say R , and both b and c must be conjugate to every inertia line passing through O and lying in R .

Thus if g_1 and g_2 be the two generators of R which pass through O , no element of either b or c with the exception of O is either *before* or *after* any element of either g_1 or g_2 .

Thus the optical lines g_1 and g_2 are such that g_1 and b lie in an optical plane and also g_2 and b lie in an optical plane.

The optical lines f_1' and f_2' on the other hand are such that both of them lie in an acceleration plane containing b .

NORMALITY OF GENERAL LINES HAVING A COMMON ELEMENT.

We are now in a position to define what we mean when we say that a general line a is "*normal*" to a general line b , which has an element in common with it.

Since a and b are not always general lines of the same kind, it is difficult to give any simple definition which will include all cases; but the introduction of the word "*normal*" is justified by the simplification which is thereby brought about in the statement of many theorems.

Only one case will be found to be strictly analogous to the normality of intersecting straight lines in ordinary geometry; namely the case of two separation lines.

The other cases are so different from our ordinary ideas of lines "*at right angles*" that we do not propose to use this expression in connection with them.

Thus for instance any optical line is to be regarded as being "*normal to itself*," and the use of the words "*at right angles*" would, in this case, clearly be an abuse of language.

The extension of the idea of normality from the cases of general lines having a common element to the cases of general lines which have not a common element is however quite analogous to the corresponding extension in ordinary geometry and will be made subsequently.

We are at present only concerned with the cases of general lines having a common element and shall naturally include among these that of an optical line being "*normal to itself*."

Thus the complete definition of the normality of general lines having a common element is to be taken as consisting of the following four particular definitions A , B , C and D .

Definition A. An optical line will be said to be *normal to itself*.

Definition B. If an optical line a and a separation line b have an element O in common and if no element of b with the exception of O be either *before* or *after* any element of a , then b will be said to be *normal* to a and a will be said to be *normal* to b .

Definition C. If an inertia line a and a separation line b be conjugate one to the other, then a will be said to be *normal* to b and b will be said to be *normal* to a .

Definition D. A separation line a having an element O in common with a separation line b will be said to be *normal* to b provided an acceleration plane P exists containing b and such that every inertia line in P which passes through O is conjugate to a .

In this last case, since there is one single inertia line in P which passes through O and is conjugate to b , it is evident that a and b must lie in a separation plane.

If c be this inertia line then, by Theorem 132, every inertia line which passes through O and lies in the acceleration plane containing c and a is conjugate to b and so b satisfies the definition of being normal to a .

Let the separation plane containing a and b be denoted by S .

Then c is conjugate to both a and b and therefore is conjugate to every separation line in S which passes through O .

It follows, by Theorem 131, that there is one and only one separation line in S and passing through O which is conjugate to every inertia line in P which passes through O and the separation line a has this property.

Now it is easy to see that a is the only separation line in S and passing through O which is normal to b ; for suppose, if possible, that a' is another such separation line.

Then, by the definition, there must exist an acceleration plane, say P' , containing b and such that every inertia line in P' which passes through O is conjugate to a' .

Then there would exist one single inertia line, say c' , through O and lying in P' which would be conjugate to b .

Thus c' would be conjugate to every separation line in S which passed through O and therefore would be conjugate to a .

But now P' could not be identical with P , for, as we have seen, a is the only separation line in S and passing through O which is conjugate to every inertia line in P which passes through O and a' has been supposed distinct from a .

But, by Theorem 134, it follows that a must be conjugate to every inertia line in P' which passes through O .

Thus we should have two distinct separation lines a and a' both lying in S and passing through O and both conjugate to every inertia line in P' which passes through O .

But this is impossible by Theorem 131 and so the assumption of the existence of two distinct separation lines in S which pass through O and are normal to b leads to a contradiction and therefore is not true.

Thus there is one and only one separation line in S which passes through O and is normal to b .

Again since b lies in P while a cannot lie in P it follows that if a separation line a be normal to a separation line b having an element in common with it, then a and b must be distinct.

If b be any general line in an acceleration plane P and O be any element of b , then we know that if b be either an inertia or separation line there is one and only one general line through O and lying in P which is conjugate and therefore *normal* to b .

Also, from our definitions, if b be an optical line there is still one and only one general line through O and lying in P which is normal to b : namely b itself.

Thus we have the following general result:

If P be either a separation plane or an acceleration plane and if b be any general line in P and O be any element in b , then there is one and only one general line lying in P and passing through O which is normal to b .

Now we have seen that if a separation line a be normal to a separation line b having an element in common with it, then a and b lie in a separation plane.

Thus two intersecting separation lines in an optical plane cannot be normal one to another.

Any separation line, however, which lies in an optical plane is normal to every optical line in the optical plane since no element of the separation line except the element of intersection is either *before* or *after* any element of any optical line in the optical plane.

Since there is one and only one optical line which passes through any element of an optical plane and lies in the optical plane we have the following result:

If P be an optical plane and if b be any separation line in P and O be any element in b , then there is one and only one general line lying in P and passing through O which is normal to b .

If on the other hand b be an optical line lying in P , then every general line in P which passes through O (including b itself) is normal to b .

We have now to prove the general theorem that: *if b and c be two distinct general lines having an element O in common and if a general line a passing through O be normal to both b and c , then a is normal to every general line which passes through O and lies in the general plane containing b and c .*

We have already proved a number of special cases of this general theorem.

(1) If b and c be both optical lines and a be a separation line, then b and c lie in an acceleration plane, say P , and if O' be any element of a distinct from O there will be an acceleration plane, say P' , passing through O' and parallel to P .

Then O and O' will be representatives of one another in the parallel acceleration planes P and P' and so, by Theorem 101, a is conjugate to every inertia line in P which passes through O .

Thus a is normal to every separation line in P which passes through O , to every inertia line in P which passes through O and to every optical line in P which passes through O .

(2) If b and c be both inertia lines and a be a separation line, the same result follows from Theorem 102.

(3) If b be an optical line and c an inertia line while a is a separation line, the same result follows from Theorem 130.

(4) If b be a separation line and c an inertia line while a is a separation line, it follows by Theorem 134 that a must be normal to every inertia line which passes through O and lies in the acceleration plane containing b and c .

Thus as before a must be normal to every general line which passes through O and lies in the acceleration plane.

(5) If b and c be both separation lines and a an inertia line, then as we have seen b and c must lie in a separation plane and as was shown in Theorem 122 a is conjugate and therefore normal to every separation line passing through O and lying in this separation plane.

(6) If b be an optical line and c a separation line while a is identical with b , then as we have already seen b and c lie in an optical plane while a is normal to every general line which passes through O and lies in this optical plane.

Several other cases remain to be considered and these form the subject of Theorems 135 to 137.

We shall postpone the enumeration of the various remaining cases till we have proved these Theorems.

THEOREM 135.

If a separation line c be normal to a separation line b which it intersects in the element O and if further c be normal to an optical line a' which it also intersects in the element O , then c is normal to every general line passing through O and lying in the general plane containing b and a' .

By the definition of normality there exists an acceleration plane, say P , containing b and such that every inertia line in P which passes through O is conjugate to c .

In case a' should lie in this particular acceleration plane the result follows directly and so we shall suppose that a' does not lie in P .

We shall therefore suppose that a' and b lie in a general plane P' distinct from P .

From the remarks at the end of Theorem 134 it is evident that P' may be either an acceleration plane or an optical plane.

The mode of proof is similar to that employed in Theorem 134 except that a' is here an optical line instead of an inertia line.

Thus the proof that c is conjugate to every inertia line passing through O and lying in either of the three acceleration planes containing a' and OE_1 , a' and OE_2 , or a' and OC , follows in this case from Theorem 130 instead of Theorem 102.

Everything else follows exactly as in Theorem 134 and we find that, if OG be any general line in P' which passes through O and is distinct from b , then OG lies in some acceleration plane such that every inertia line in the latter which passes through O is conjugate to c .

Thus if OG be a separation line it satisfies the condition that c should be normal to it.

Also if OG should be either an optical line or an inertia line c must also be normal to it, and so the theorem is proved.

REMARKS.

From the definition of the normality of intersecting separation lines it is evident that we may have a separation line normal to two (or more) separation lines in an acceleration plane.

From the last theorem it is also evident that we may have a separation line normal to two (or more) separation lines in an optical plane.

We may also have a separation line normal to two (or more) separation lines in a separation plane, as may easily be seen from the following considerations :

In the remarks at the end of Theorem 133 it was pointed out that we may have an acceleration plane and a separation plane having only

one element in common and such that each inertia line through the common element in the former is conjugate to every separation line through it in the latter.

Let P be the acceleration plane, S the separation plane and O the common element.

Let a and b be any two separation lines passing through O and lying in S , and let c be any separation line passing through O and lying in P .

Then a satisfies the definition of being normal to c and therefore c is normal to a .

Similarly c must be normal to b .

Thus c is normal to the two separation lines a and b which lie in the separation plane S .

THEOREM 136.

If three distinct separation lines a , b and c have an element O in common and if c be normal to both a and b , then c is normal to every general line which passes through O and lies in the general plane containing a and b .

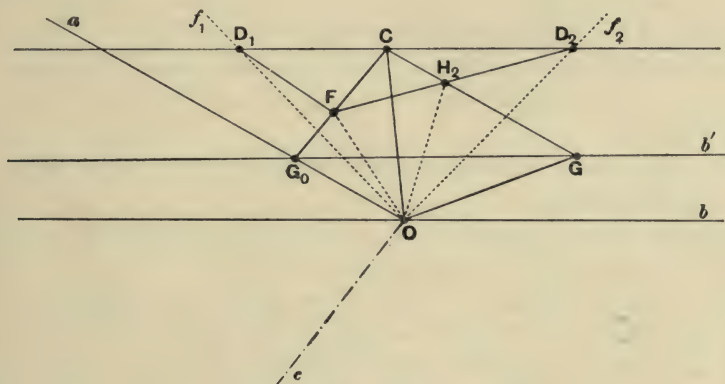


Fig. 36.

By the definition of the normality of intersecting separation lines there must exist an acceleration plane, say P , containing b and such that every inertia line in P which passes through O is conjugate to c .

Let f_1 and f_2 be the two generators of P which pass through O and let D_1 be any element in f_1 which is *after* O .

Let the separation line through D_1 parallel to b intersect f_2 in D_2 .

Then D_2 must also be *after* O .

Let C be any element linearly between D_1 and D_2 .

Then by Theorem 129 OC is an inertia line and C is *after* O .

But c is normal to the inertia line OC and to the separation line a and therefore by case (4) on page 207 c must be normal to every inertia line (and therefore also every general line) which passes through O and lies in the acceleration plane containing OC and a .

Let R be this acceleration plane.

If R should coincide with P the result follows directly and so we shall suppose that R is distinct from P .

Let S be the general plane containing a and b .

Then S will be distinct from both P and R , and, as was pointed out in the remarks at the end of the last theorem, S may be an acceleration plane, an optical plane, or a separation plane.

Let one of the generators of R which pass through C intersect a in G_0 and let the generator of the opposite set passing through O intersect CG_0 in F .

Then since O does not lie in the optical line CG_0 but is *before* the element C of it, it follows that F must lie in the α sub-set of O and therefore F is *after* O .

Let b' be the general line through G_0 parallel to b .

Then since G_0 lies in S it follows that b' lies in S .

Let Q be the general plane containing b' and G_0C .

Then since D_1D_2 is parallel to b and is distinct from b' it follows that it is parallel to b' , and since D_1D_2 passes through C it must lie in the general plane Q .

Thus D_1 , D_2 and F are three distinct elements in Q which do not all lie in one general line.

Now any general line in S which passes through O and is distinct from b must intersect b' in some element, say G .

Then the general line CG lies in Q and is distinct from D_1D_2 .

If then G does not coincide with G_0 it follows, by Theorem 128, that CG must either intersect D_1F in an element H_1 linearly between D_1 and F , or else must intersect D_2F in an element H_2 linearly between D_2 and F .

But since O is *before* both D_1 and F it follows by Theorem 129 that OH_1 is an inertia line and similarly since O is *before* both D_2 and F it follows that OH_2 is an inertia line.

Now c is normal to every general line in P which passes through O and also to every general line in R which passes through O and therefore c is normal to the three optical lines OD_1 , OD_2 and OF .

Thus c must be conjugate to every inertia line which passes through O and lies either in the acceleration plane containing OD_1 and OF , or the acceleration plane containing OD_2 and OF .

Thus c is conjugate to OH_1 and also to OH_2 .

But c is conjugate to OC and therefore is conjugate to every inertia line which passes through O and lies in the acceleration plane containing OC and OH_1 or the acceleration plane containing OC and OH_2 .

Thus since OG lies in the acceleration plane containing OC and OH_1 or in the acceleration plane containing OC and OH_2 as the case may be, it follows that c must be normal to OG .

Thus, including the separation lines a and b , the separation line c is normal to every general line which passes through O and lies in the general plane S .

THEOREM 137.

If two distinct separation lines a and b intersect in an element O and if an optical line c passing through O be normal to both a and b , then c is normal to every general line which passes through O and lies in the general plane containing a and b .

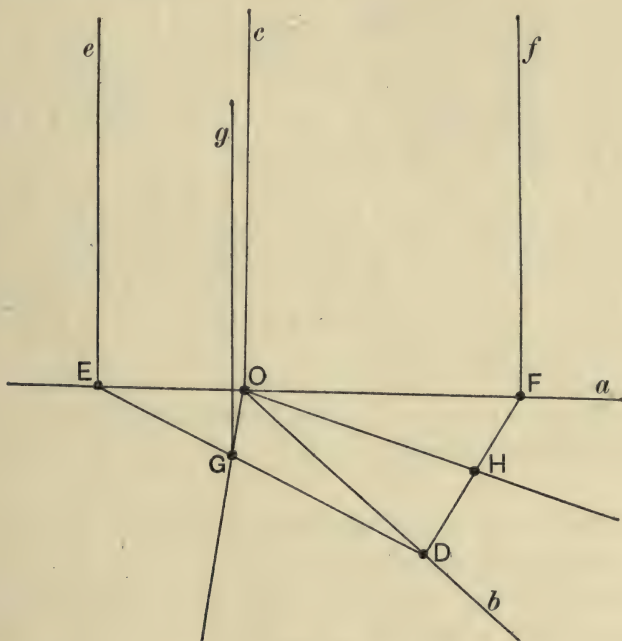


Fig. 37.

From the definition of the normality of an optical line to an intersecting separation line it follows that c and a lie in an optical plane, say P , while c and b lie in an optical plane, say Q .

If P should be identical with Q we already know that c is normal to

every general line in P which passes through O including the optical line c itself.

Let us suppose next that P is distinct from Q .

We have already seen that in this case a and b lie in a separation plane, say S , and further we have seen that no element of b with the exception of O is either *before* or *after* any element of P .

Let D be any element of b distinct from O and let E be any element of a distinct from O , while F is any element of a such that O is linearly between E and F .

Let e and f be optical lines through E and F respectively and parallel to c .

Then D is neither *before* nor *after* any element either of e or of f and so by Theorem 45 no element of DE with the exception of E is either *before* or *after* any element of e , and no element of DF with the exception of F is either *before* or *after* any element of f .

But now by Theorem 128 any general line passing through O and lying in S and which is distinct from both a and b must either intersect DE in some element, say G , linearly between D and E or else must intersect DF in some element, say H , linearly between D and F .

Thus G is neither *before* nor *after* any element of e while H is neither *before* nor *after* any element of f .

If then g be an optical line through G parallel to e it will be a neutral-parallel and since c is a neutral-parallel of e and G does not lie in c it follows by Theorem 27 that g is a neutral-parallel of c .

Thus G is neither *before* nor *after* any element of c and therefore, by Theorem 45, no element of OG with the exception of O is either *before* or *after* any element of c .

Thus c is normal to OG and similarly it is normal to OH .

It follows that c is normal to every general line which passes through O and lies in S , and so the theorem is proved.

ENUMERATION OF CASES OF GENERAL THEOREM CONTINUED.

We now resume the enumeration of the various cases of the general theorem stated on page 207 and which was interrupted in order to prove Theorems 135 to 137.

Six cases have already been mentioned and we now proceed with case (7).

(7) If b be a separation line and c an optical line while a is a separation line and if b and c lie in an acceleration plane, the result follows from Theorem 135.

(8) If b and c be both separation lines lying in an acceleration plane and if a be also a separation line, the result follows from Theorem 136.

(9) If b be a separation line and c an optical line while a is a separation line and if b and c lie in an optical plane, the result follows from Theorem 135.

(10) If b and c be both separation lines lying in an optical plane and if a be also a separation line, the result follows from Theorem 136.

(11) If b and c be both separation lines lying in an optical plane and if a be an optical line also in the optical plane, the result still holds as was pointed out at the beginning of Theorem 137.

(12) If b and c be both separation lines lying in a separation plane and if a be also a separation line, the result follows from Theorem 136.

(13) If b and c be both separation lines lying in a separation plane and if a be an optical line, the result follows from Theorem 137.

If now we combine cases (1), (2), (3), (4), (7) and (8) we see that b and c may be any two intersecting general lines in an acceleration plane taking a as a separation line.

If we combine cases (9) and (10) we see that b and c may be any two intersecting general lines in an optical plane taking a as a separation line.

Further combining cases (6) and (11) we also see that b and c may be any two intersecting general lines in an optical plane taking a as an optical line.

Finally from cases (12), (13) and (5) we see that b and c may be any two intersecting general lines in a separation plane taking a as a separation line, an optical line, or an inertia line.

Thus for all the different possible cases of the normality of general lines having a common element this general result holds.

THEOREM 138.

If b and c be two separation lines intersecting in an element O and lying in a separation plane S and such that c is normal to b , then if O' be any other element of b , the normal to b through O' in the separation plane S is parallel to c .

From the definition of the normality of intersecting separation lines it follows that there must exist an acceleration plane P containing b and such that every inertia line in P which passes through O is conjugate to c .

Let a_1 and a_2 be any two such inertia lines and let a_1' and a_2' be inertia lines passing through O' and parallel to a_1 and a_2 respectively.

Let c' be a separation line passing through O' and parallel to c .

Then c' will lie in S .

But by Theorem 107 both a_1' and a_2' must be conjugate to c' and so by Theorem 102 c' is conjugate to every inertia line in the acceleration plane containing a_1' and a_2' which passes through the element O' .

But this acceleration plane is the acceleration plane P which contains the separation line b and so c' satisfies the definition of being normal to b .

Further c' passes through O' and lies in S and we have already seen that there is only one normal to b which satisfies these conditions.

Thus the normal to b through O' in the separation plane S is parallel to c as was to be proved.

THEOREM 139.

If b and c be two separation lines intersecting in an element O and such that c is normal to b and if b' and c' be two other separation lines intersecting in an element O' and respectively parallel to b and c , then c' is normal to b' .

Since c is normal to b there must exist an acceleration plane P containing b and such that every inertia line in P which passes through O is conjugate to c .

Let a_1 be one such inertia line which we shall suppose does not also pass through O' .

Then through O' there is an inertia line, say a_1' , which is parallel to a_1 .

Thus b' and a_1' determine an acceleration plane P' which will be either identical with P or parallel to P according as O' does or does not lie in P .

Let a_2 be a second inertia line in P and passing through O but not through O' .

Then through O' there is an inertia line, say a_2' , parallel to a_2 and lying in P' .

Then by Theorem 107 both a_1' and a_2' are conjugate to c' and so by Theorem 102 c' is conjugate to every inertia line in P' which passes through O' .

But P' contains b' and so c' satisfies the definition of being normal to b' .

THEOREM 140.

If an optical line b intersects a separation line c in an element O and if c be normal to b and if further b' and c' be an optical line and a separation line respectively which intersect in an element O' and are respectively parallel to b and c , then c' will be normal to b' .

From the definition of the normality of a separation line to an optical line it follows that b and c lie in an optical plane, say P .

Further b' and c' lie in a general plane P' which must be either identical with P or parallel to P according as O' does or does not lie in P .

In either case P' is an optical plane and accordingly, since b' is an optical line and c' a separation line, it follows that c' must be normal to b' .

REMARKS.

By combining Theorems 107, 139 and 140 we obtain the general result that *if b and c be two general lines intersecting in an element O and such that the one is normal to the other and if b and c be two other general lines intersecting in an element O' and respectively parallel to b and c , then of these latter two general lines the one is normal to the other.*

If now we remember that an optical line is to be regarded as *normal to itself*, we are in a position to extend the definition of the normality of general lines to the case of general lines which have no element in common, as is done with straight lines in ordinary geometry.

Definition. A general line b will be said to be *normal* to a general line c' which has no element in common with it, provided that a general line b' taken through any element of c' parallel to b is normal to c' in the sense already defined.

It is evident from the above considerations that, in these circumstances, if a general line c be taken through any element of b parallel to c' then c will be normal to b and so c' will be normal to b .

Further we have the result that *any two parallel optical lines are to be regarded as normal to one another.*

Again, if P be an acceleration plane and if a be any general line in P and A be any element in P , then there is one single general line in P and passing through A which is normal to a .

If however a be an optical line, the normal to a through A is either identical with a or parallel to it according as A does or does not lie in a .

If, on the other hand, P be an optical plane, there is one single general line in P and passing through A which is normal to a , except when a is an optical line, in which case every general line in P which passes through A is normal to a .

If P be a separation plane there is one single general line in P which passes through A and is normal to a and in this case the normal to a always intersects a as in ordinary geometry.

Definition. A general line a will be said to be *normal* to a general plane P provided a be normal to every general line in P .

It is evident that if a general line a be normal to two intersecting general lines in a general plane P , then a will be normal to P .

In case P be an optical plane it is clear that, according to the above definition, any generator of P is normal to P .

This is the only case in which a general line can be normal to a general plane which contains it.

In no other case can a general line which is normal to a general plane have more than one element in common with the latter.

As was pointed out in the remarks at the end of Theorem 133 we may have an acceleration plane and a separation plane having only one element in common and such that each inertia line through the common element in the former is conjugate to every separation line through it in the latter.

It is evident now that we have here two general planes which are so related that any general line in the one is normal to any general line in the other.

In ordinary *three-dimensional* geometry two planes cannot be so related, and when we speak of one plane being normal to another the normality is not of this complete character.

We shall therefore introduce the following definition:

Definition. If two general planes be so related that every general line in the one is normal to every general line in the other, the two general planes will be said to be *completely normal* to one another.

THEOREM 141.

If P be an acceleration plane and O be any element in it, there is at least one separation plane passing through O and completely normal to P .

Let P_1 be any acceleration plane which is parallel to P and let O_1 be the representative of O in P_1 .

Then, by Theorem 101, the separation line OO_1 is conjugate to any inertia line in P which passes through O and so OO_1 is normal to P .

Let a_1 and a_2 be the two generators of P which pass through O , and let OO_1 be denoted by b_1 .

Then a_1 and b_1 lie in one optical plane, say Q_1 , while a_2 and b_1 lie in another optical plane, say Q_2 .

Now by Post. XIX there is at least one element, say A , which is neither *before* nor *after* any element of Q_1 .

Thus through A there is an optical line, say a_1' , which is neutrally parallel to a_1 and so a_1 and a_1' lie in an optical plane, say R_1 , which is distinct from Q_1 .

Again if P_2 be an acceleration plane through A parallel to P it will contain a_1' .

Let O_2 be the representative of O in P_2 .

Then O_2 must lie in a_1' and so OO_2 must lie in the optical plane R_1 .

But OO_1 lies in Q_1 while OO_2 lies in R_1 , and Q_1 and R_1 have only the optical line a_1 in common.

Thus since OO_1 and OO_2 are both separation lines they must be distinct.

Now by Theorem 101 OO_2 is conjugate to any inertia line in P which passes through O , and so OO_2 is normal to P .

Let OO_2 be denoted by b_2 .

Then no element of b_2 is either *before* or *after* any element of b_1 , and since b_1 and b_2 have the element O in common, they must lie in a separation plane, say S .

Thus any inertia line in P which passes through O is conjugate to both b_1 and b_2 and therefore also conjugate to every separation line in S which passes through O .

Thus every general line in P is normal to every general line in S and so the separation plane S is completely normal to P .

Thus, since S passes through O , the theorem is proved.

THEOREM 142.

If P be a separation plane and O be any element in it, there is at least one acceleration plane passing through O and completely normal to P .

If we take any two separation lines in P and passing through O , then by Theorem 106 there is at least one inertia line, say a_1 , which is conjugate to both of them and therefore is normal to P .

Let b_1 be any separation line in P which passes through O and let Q be the acceleration plane containing a_1 and b_1 .

Then by Theorem 131 there is one and only one separation line in P and passing through O which is conjugate to every inertia line in Q which passes through O .

Let b_2 be this separation line.

Then as was remarked at the end of Theorem 133 b_2 is conjugate to certain other inertia lines passing through O which do not lie in Q .

Let a' be any such inertia line and let Q' be the acceleration plane containing a' and b_1 .

Then, by Theorem 134, b_2 is conjugate to every inertia line in Q' which passes through O .

Let a_2 be the one single inertia line in Q' and passing through O which is conjugate to b_1 and let R be the acceleration plane containing a_1 and a_2 .

Then a_1 and a_2 are each conjugate to both b_1 and b_2 .

Thus both a_1 and a_2 are conjugate to every separation line in P which passes through O and so every separation line in P which passes through O is conjugate to every inertia line in R which passes through O .

Thus every general line in P is normal to every general line in R and so the acceleration plane R is completely normal to P .

Thus, since R passes through O , the theorem is proved.

THEOREM 143.

If P be an optical plane and O be any element in it, there is at least one optical plane passing through O and completely normal to P .

Let a be the generator of P which passes through O and let b be any separation line in P which passes through O .

Then, by Post. XIX, there is at least one element, say A , which is neither *before* nor *after* any element of P .

The general line OA is thus a separation line and, by Theorem 45, no element of OA with the exception of O is either *before* or *after* any element of a .

Thus a is normal to OA and it is also normal to b and so, since OA and b must lie in a separation plane, say S , it follows that the optical line a is normal to S .

But now we know that there is one single separation line, say c , which passes through O , lies in S and is normal to b .

Then c is normal to both a and b and therefore is normal to P .

But c and a lie in an optical plane which is distinct from P and which we shall call R .

Further a is an optical line in P and therefore is normal to P .

Thus any general line in P is normal to the two intersecting general lines a and c which lie in R and so every general line in P is normal to every general line in R .

It follows that R is completely normal to P and, since R passes through O , the theorem is proved.

REMARKS.

By combining Theorems 141, 142 and 143 we get the general result:

If P be any general plane and O be any element in it, there is at least one general plane passing through O and completely normal to P .

If R be this general plane which is completely normal to P and if O' be any element not lying in P , then O' either may or may not lie in R .

If O' does not lie in R then there is a general plane, say R' , passing through O' and parallel to R .

It is evident that since R is completely normal to P we must also have R' completely normal to P and so we may generalize the above result and say:

If P be any general plane and O be any element whatever, there is at least one general plane passing through O and completely normal to P .

Let O be any element and let S be any separation plane passing through O , while P is an acceleration plane also passing through O and completely normal to S .

Let a be any separation line in S which passes through O and let b be the one single separation line in S and passing through O which is normal to a .

Let c be any separation line passing through O and lying in P and let d be the one single inertia line in P and passing through O which is normal to c .

Then both c and d are normal to both a and b and so we have the three separation lines a , b and c all passing through O and each of them normal to the other two; while in addition to these we have the inertia line d also passing through O and normal to all three.

This result marks an important stage in the development of our theory as it suggests the possibility of setting up a system of coordinate axes one of which axes is of a different character from the remaining three.

Another important result is the following:

If S be a separation plane and if P be an acceleration plane passing through any element O of S and completely normal to S , then there are

two generators of P which pass through O and each of them is normal to the separation plane S .

Thus there are at least two optical lines which pass through any element of a separation plane and are normal to it.

THEOREM 144.

If P be an acceleration or separation plane and O be any element which does not lie in it, there is one single general line passing through O and normal to P which has an element in common with P .

We already know that if a be a separation line and if O be any element which does not lie in it, then, in whatever type of general plane O and a may lie, there is one single general line passing through O and lying in this general plane which is normal to a .

Further, if d be this general line normal to a , then d must intersect a in some element, say A .

Now suppose that a lies in the acceleration or separation plane P .

Then there is one single general line passing through A and lying in P which is normal to a .

Let b be this general line.

Then since P is an acceleration or separation plane and a is a separation line, b must be distinct from a and must be either an inertia or separation line and cannot be an optical line.

Now we know that in whatever type of general plane O and b may lie there is one single general line passing through O and lying in this general plane which is normal to b .

Let c be this general line.

Then since b is not an optical line this normal to it through O cannot be parallel to b and therefore must intersect b in some element, say B .

Now a is normal to the two general lines d and b which intersect in A and accordingly a is normal to every general line in the general plane containing d and b and therefore is normal to c .

But c is normal to the two intersecting general lines a and b which lie in P and therefore c is normal to P .

Since c has the element B in common with P we have proved that there is at least one general line through O and normal to P which has an element in common with P .

It remains to show that there is only one general line having this property.

Consider first the case where P is a separation plane and let B' be any element in P distinct from B .

Then BB' is a separation line and so in whatever type of general plane O and BB' may lie there is one single general line passing through O lying in this general plane and normal to BB' .

But OB passes through O and is normal to BB' and therefore OB' cannot be normal to BB' and so cannot be normal to P .

This proves that OB is the only general line through O and normal to P which has an element in common with P provided P be a separation plane.

This method does not serve if P be an acceleration plane since BB' might in this case be an optical line.

If P be an acceleration plane, let P' be an acceleration plane passing through O and parallel to P .

Then O and B must be representatives of one another in the parallel acceleration planes P' and P .

If B' were any other element in P distinct from B and such that OB' were normal to P , then B' would be also the representative of O in P which we know is impossible.

Thus again OB is the only general line through O and normal to P which has an element in common with P .

The theorem thus holds for both separation and acceleration planes.

THEOREM 145.

If P be an optical plane and O be any element which does not lie in it, then:

(1) *If O be neither before nor after any element of P there is one single generator of P such that every general line which passes through O and intersects this generator is normal to P .*

(2) *If O be either before or after any element of P there is no general line passing through O and having an element in common with P which is normal to P .*

As regards the first part of this theorem, if we carry out the construction of Theorem 144 taking a as a separation line, then since P is an optical plane the general line b must be an optical line since it is normal to a .

Since O is neither *before* nor *after* any element of P it is neither *before* nor *after* any element of b .

If then OB be any general line passing through O and intersecting b in the element B , it follows by Theorem 45 that no element of OB with the exception of B is either *before* or *after* any element of b .

It follows that OB is normal to b .

But, as in Theorem 144, OB is normal to a and thus OB is normal to the two intersecting general lines a and b which lie in P and therefore it is normal to P .

Again if B' be any element in P which does not lie in b , then BB' is a separation line and so as in Theorem 144 OB' cannot be normal to P .

Thus all general lines through O normal to P which have an element in common with P intersect b .

Thus the first part of the theorem is proved.

Suppose next that O is *before* some element, say E , in P .

Then through E one single generator of P passes which we may denote by f .

Since O does not lie in f but is *before* an element of f , it follows that through O there is an optical line which is a before-parallel of f and which we shall denote by c .

If f' be any other generator of P it will be a neutral-parallel of f and so by Theorem 25 (a) c will be a before-parallel of f' .

Thus O is *before* elements of every generator of P .

Similarly if O be *after* any element of P it is *after* elements of every generator of P .

Thus in case O be either *before* or *after* any element of P it will lie in an acceleration plane along with any selected generator of P .

Let OB be any general line passing through O and having the element B in common with P and let b be the generator of P which passes through B .

Then OB and b lie in an acceleration plane and intersect in B and so since b is an optical line OB cannot be normal to it.

Thus OB cannot be normal to P and therefore there is in this case no general line passing through O and having an element in common with P which is normal to P .

THEOREM 146.

If a general line d have an element A in common with a general plane P , there is at least one general line passing through A and lying in P which is normal to d .

If d lies completely in P we already know that the theorem holds and so we shall suppose that A is the only element common to d and P .

We shall first consider the case where P is an acceleration or separation plane.

In this case, if O be any element of d distinct from A , there is, by Theorem 144, one single general line passing through O and normal to P which has an element in common with P .

Let B be this element.

If B should coincide with A then every general line passing through A and lying in P would be normal to d .

If B does not coincide with A let a be the one single general line passing through A and lying in P which is normal to AB .

Then since OB is normal to P it must be normal to a .

Thus a is normal to the two intersecting general lines AB and OB and therefore is normal to the general plane containing them.

Thus the general line a must be normal to d and since a passes through A and lies in P the theorem is proved for the case where P is an acceleration or separation plane.

Suppose next that P is an optical plane and let b be the generator of P which passes through A .

Now since b is an optical line it follows that the intersecting general lines b and d must lie in a general plane, say Q , which must be either an optical plane or an acceleration plane.

Suppose first that Q is an optical plane.

Then since b is an optical line in Q and d intersects b , it follows that d must be a separation line and b must be normal to d .

But b passes through A and lies in P and so the theorem is proved for this case.

Next consider the case where Q is an acceleration plane.

Let b' be any generator of P distinct from b .

Then since b' is a neutral-parallel of b it follows that an acceleration plane Q' through any element of b' and parallel to Q will contain b' .

If then A' be the representative of A in Q' , the general line AA' will be normal to Q and therefore will be normal to d .

But, since b' is neutrally parallel to b which contains the element A , the element A' must lie in b' and therefore in the optical plane P .

Thus the general line AA' must lie in P and, since it passes through A and is normal to d , the theorem holds also in this case.

Thus the theorem holds in general.

THEOREM 147.

If three general lines a , b , and c have an element O in common, there is at least one general line passing through O which is normal to all three.

If we take any two of the three given general lines, say a and b , it

follows, since they have the element O in common, that they lie in a general plane, say P .

Then by Theorems 141, 142, and 143 there is at least one general plane passing through O and completely normal to P .

Let Q be this general plane.

Then, since c has the element O in common with Q , it follows, by Theorem 146, that there is at least one general line, say d , passing through O and lying in Q which is normal to c .

But since d lies in Q it is normal to both a and b and thus is normal to all three general lines.

Thus the theorem is proved.

Definition. If a general line and a general plane have one single element in common, they will be said to *intersect* in that element.

Definition. If a general line a and a general plane P intersect, then the aggregate of all elements of P and of all general planes parallel to P which intersect a will be called a *general threefold*.

It will be found that, just as there are three types of general line and three types of general plane, so there are three types of general threefold.

In the case of general threefolds however, unlike that of general lines or of general planes, we are able to give a definition which applies to all three types without first considering any of the special cases.

From the definition it is clear that if a general threefold W be determined by a general line a intersecting a general plane P then any other general plane P' parallel to P and intersecting a may take the place of P , so that a and P' will also serve to determine W .

Again if a intersects P in the element O and if a' be a general line parallel to a and intersecting P in another element O' , then a and a' will lie in a general plane, say Q .

If through any element O_1 of a distinct from O the general plane P_1 passes parallel to P , then by Theorem 124 the general plane Q must have a second element in common with P_1 .

Thus P_1 and Q have a general line in common which must be parallel to OO' and so the general line a' must intersect P_1 in some element O'_1 .

Thus a' intersects every general plane parallel to P which intersects a and similarly a intersects every general plane parallel to P which intersects a' .

It follows that every element of a' lies in the general threefold determined by a and P , and also: that a' and P determine the same general threefold as a and P .

THEOREM 148.

If two distinct elements of a general line lie in a general threefold then every element of the general line lies in the general threefold.

Let the general threefold W be determined by a general plane P and a general line a which intersects it.

Let X_1 and X_2 be two distinct elements of a general line b and let them both lie in W .

If X_1 and X_2 should both lie in P or in any one of the general planes which intersect a and are parallel to P , then the general line b will lie in that general plane and therefore every element of b must lie in W .

We shall next suppose that X_1 lies in one of the set of parallel general planes, say P_1 , while X_2 lies in another, say P_2 .

Then b either may or may not lie in a general plane containing a .

Suppose first that b lies in a general plane Q along with a .

Then we may have either:

- (1) b identical with a ,
- or (2) b parallel to a ,
- or (3) b intersecting a .

If b be identical with a the result is obvious.

If b be parallel to a then, as we have already shown, every element of b lies in W .

If b intersects a then at least one of the elements X_1 , X_2 must be distinct from the element of intersection of b and a .

We may suppose that X_1 is distinct from this element of intersection.

Then the element in which a intersects P_1 must be distinct from X_1 and so the general plane Q has two distinct elements in common with P_1 .

Further since the general line a intersects all the general planes parallel to P_1 whose elements along with the elements P_1 make up W , it follows, by Theorem 124, that Q has a general line in common with each of these general planes and all these general lines are parallel to one another.

Now since b does not lie in P_1 it follows that b must intersect all these general planes and similarly a general plane through any element of b distinct from X_1 and taken parallel to P_1 must intersect a .

Thus we see that in this case also every element of b lies in W and further that b and P_1 determine the same general threefold as a and P_1 : namely W .

Thus the theorem holds provided b and a lie in one general plane.

Finally suppose as before that X_1 lies in P_1 and X_2 in P_2 and that b and a do not lie in one general plane.

Let a intersect P_1 in the element Y_1 and let b' be a general line through Y_1 parallel to b .

Then b and b' lie in a general plane, say R , which has the two elements X_1 and Y_1 in common with P_1 and has the element X_2 in common with the parallel general plane P_2 .

Thus R has a general line in common with P_2 which is parallel to X_1Y_1 and so b' must intersect P_2 in some element, say Y_2 .

But now, from what we have already proved, every element of b' must lie in W and also b' and P_2 determine the general threefold W equally with a and P_2 or a and P .

Again since b is parallel to b' it follows from what we have already proved that every element of b lies in the general threefold determined by b' and P_2 : that is in W ; and that b and P_2 may also be taken as determining the general threefold W .

Thus the theorem holds in general.

REMARKS.

It is evident from the above that if a general threefold W be determined by a general plane P and a general line a which intersects P , then a general line b which has two distinct elements in common with W , which do not both lie in P or do not both lie in one of the general planes parallel to P and intersecting a , will intersect all these general planes including P .

Further b and P , or b and any one of these general planes, will also determine W .

Again if a general plane Q have two distinct elements X_1 and X_2 in common with W , then Q will have at least one general line in common with W : namely the general line X_1X_2 since, by the above theorem, every element of X_1X_2 must lie in W , and we already know that every element of it must also lie in Q .

It is not however possible from this to prove that Q and W have more than one general line in common.

THEOREM 149.

If a general plane have three distinct elements in common with a general threefold and if these three elements do not all lie in one general line then every element of the general plane lies in the general threefold.

Let the general threefold W be determined by a general plane P and a general line a which intersects P .

Let X_1 , X_2 and X_3 be three distinct elements of a general plane Q which do not all lie in one general line and suppose that X_1 , X_2 and X_3 all lie in W .

If all these three elements should lie in P or if they should all lie in one of the general planes parallel to P which intersect a , then Q would be identical with the general plane in which they all lie and accordingly every element of Q would lie in W .

If X_1 , X_2 and X_3 do not all lie in one of this set of general planes, suppose that X_1 lies in the general plane P_1 of the set while X_2 lies in another distinct general plane of the set, say P_2 .

Then X_3 will lie in some general plane P_3 of the set which may be either identical with P_1 or with P_2 , or may be distinct from both.

Now since X_1 and X_2 lie in two distinct general planes of the set it follows that the general line X_1X_2 intersects every general plane of the set and therefore must intersect P_3 in some element, say O .

Further, since X_1 , X_2 and X_3 do not all lie in one general line, it follows that X_3 and O must be distinct elements.

Thus the general planes P_3 and Q have two distinct elements X_3 and O in common and therefore have the general line OX_3 in common which accordingly lies in W .

Again the general threefold W , as we have seen, may be determined by the general plane P_3 and the general line X_1X_2 which intersects P_3 in O .

But now every element of Q lies either in X_1X_2 or in a general line parallel to X_1X_2 and intersecting OX_3 .

We have however seen that every element of any such general line must lie in W .

It follows that every element of Q must lie in W .

Thus the theorem holds in all cases.

THEOREM 150.

(1) *If a general line b lies in a general threefold W and if A be any element lying in W but not in b , then the general line through A parallel to b also lies in W .*

(2) *If a general plane P lies in a general threefold W and if A be any element lying in W but not in P , then the general plane through A parallel to P also lies in W .*

The first part of the theorem may be proved as follows:

The general line b and the element A determine a general plane, say Q , having three elements in common with W which do not all lie in one general line, and so, by Theorem 149, Q lies in W .

But the general line through A parallel to b must lie in Q and therefore must lie in W .

This proves the first part of the theorem.

In order to prove the second part let b and c be two intersecting general lines which both lie in P and therefore in W .

The element A does not lie in P and therefore cannot lie either in b or c .

If then b' and c' be general lines through A parallel to b and c respectively, it follows from the first part of the theorem that b' and c' both lie in W .

If then P' be the general plane containing b' and c' it will contain three distinct elements in common with W which do not all lie in one general line and so, by Theorem 149, P' must lie in W .

But P' is parallel to P and passes through A and so the theorem is proved.

THEOREM 151.

If a general threefold W be determined by a general plane P and a general line a which intersects P , then if Q be any general plane lying in W , and if b be any general line lying in W and intersecting Q , the general plane Q and the general line b also determine the same general threefold W .

It is evident from the remarks at the end of Theorem 148, this above holds in the special case where Q is one of the set of general planes consisting of P and all general planes parallel to P which intersect a .

We shall therefore consider the case where Q is distinct from any one of this set of general planes which we shall for convenience refer to as the *primary set*.

Let X_1 be any element in Q and let c_1 and c_1' be any two distinct general lines lying in Q and passing through X_1 .

Then c_1 and c_1' could not both lie in any general plane of the primary set, for if so Q would require to be identical with that general plane, contrary to hypothesis.

Thus at least one of the two general lines c_1, c_1' does not lie in any general plane of the primary set.

Suppose c_1 be a general line of this character.

Then, since Q lies in W , each element of c_1 must lie in a distinct general plane of the primary set, and c_1 must intersect every general plane of the primary set.

Let X_2 be any element of Q which does not lie in c_1 , and let c_2 be a general line through X_2 parallel to c_1 .

Then c_2 must also lie in Q and must also intersect every general plane of the primary set.

If then P' be any one general plane of the primary set, it is intersected both by c_1 and by c_2 and the elements of intersection must be distinct since c_1 and c_2 are parallel.

Thus P' has two distinct elements in common with Q and therefore has a general line in common with Q .

It follows that Q has a general line in common with each general plane of the primary set.

Now let A be any element in b other than its element of intersection with Q .

Then A must lie in some general plane of the primary set, say P_1 , since b lies in W .

Now as we have seen P_1 has a general line in common with Q , and since A does not lie in Q it cannot lie in this general line.

If then B and C be any two distinct elements in this general line, the three elements A , B and C are three distinct elements in P_1 which do not all lie in one general line.

But it is evident that A , B and C all lie in the general threefold determined by Q and b and so, by Theorem 149, the general plane P_1 must lie in this general threefold which we may call W' .

Now, since the general line c_1 intersects every general plane of the primary set, it follows from the remarks at the end of Theorem 148 that c_1 and P_1 determine the general threefold W equally with a and P .

Also since c_1 lies in Q it must lie in W' , and so, by Theorem 150, every general plane which passes through an element of c_1 and is parallel to P_1 must lie in W' .

But the general threefold W is the aggregate of all elements of P_1 and of all general planes parallel to P_1 which intersect c_1 , and so every element of W must lie in W' .

But, since Q and b both lie in W , it follows by Theorem 150 that every general plane which passes an element of b and is parallel to Q must lie in W .

Since however the general threefold W' is the aggregate of all elements of Q and of all general planes parallel to Q which intersect b , it follows that every element of W' must lie in W .

Thus the general threefolds W' and W consist of the same set of elements and are therefore identical.

Thus Q and b determine W as was to be proved.

REMARKS.

It follows directly from the above theorem that *any four distinct elements which do not all lie in one general plane determine a general threefold containing them.*

For let A, B, C, D be four distinct elements which do not all lie in one general plane.

Then no three of them can lie in one general line.

Let Q be the general plane containing A, B and C and let b be the general line DA .

Then b cannot have any other element than A in common with Q , for then D would have to lie in Q along with A, B and C contrary to hypothesis.

Thus b intersects Q .

Let W be the general threefold determined by Q and b and let W' be any general threefold containing A, B, C and D .

Then since W' contains A, B and C it follows by Theorem 149 that W' contains Q .

Also by Theorem 148, since W' contains A and D it contains b .

Thus by Theorem 151 the general threefold W' is identical with W : that is to say is identical with one definite general threefold.

Again it is clear that: *any three distinct general lines having a common element and not all lying in one general plane determine a general threefold containing them.*

THEOREM 152.

If two distinct general planes P and Q lie in a general threefold W , then if P and Q have one element in common they have a second element in common.

Let A be any element in P and let B be any element which lies in W but not in P .

Let the general line AB be denoted by a .

Then a intersects P and since it has two distinct elements in common with W it follows that a lies in W .

Then by Theorem 151, P and a may be taken as determining W and any element of W lies either in P or in a general plane parallel to P and intersecting a .

If now we call this set of mutually parallel general planes the "primary set" we have already seen in proving Theorem 151 that Q must either be identical with some general plane of the primary set or else must have a general line in common with each general plane of the primary set.

But now, since P and Q are supposed to be distinct, Q cannot be identical with P , and since Q is supposed to have an element in common with P , it follows that Q is not parallel to P .

Thus Q cannot be identical with any general plane of the primary set and therefore must have a general line in common with each of them including P .

Thus P and Q must have a second element in common.

REMARKS.

It is further evident from the above considerations that *if two distinct general planes P and Q both lie in a general threefold W , then if P and Q have no element in common they must be parallel to one another.*

Now we have already seen that we can have a separation plane S and an acceleration plane P having an element O in common and which are completely normal to one another.

We have seen that in this case P and S cannot have a second element in common.

It follows that P and S cannot lie in one general threefold.

Now let a_1 and a_2 be any two distinct general lines lying in P and passing through O .

Then S and a_1 determine a general threefold, say W_1 , while S and a_2 determine a general threefold, say W_2 .

Now W_1 and W_2 must be distinct, for if W_2 were identical with W_1 , then W_1 would contain both a_1 and a_2 and would therefore contain P .

But W_1 contains S and so this is impossible.

Thus W_1 and W_2 are distinct general threefolds each of which contains the separation plane S .

Since there are an infinite number of general lines lying in P and passing through O it follows that *there are an infinite number of general threefolds which all contain any separation plane S .*

Similarly *there are an infinite number of general threefolds which all contain any acceleration plane P .*

Without Post. XIX or some equivalent, we cannot from our remaining postulates show that there is more than one general threefold; for the proof of the existence of an acceleration plane which is completely normal to a separation plane depends upon Post. XIX.

THEOREM 153.

If a general plane P and a general line a both lie in a general threefold W and if a does not lie in P , then either a is parallel to a general line in P or else has one single element in common with P .

Let B be any element lying in P but not in a .

Then a and B determine a general plane, say Q , which must lie in W , since it contains three elements in common with W which do not all lie in one general line.

But since P and Q have the element B in common and both lie in W , therefore by Theorem 152 they have a general line in common which we may denote by b .

Since then b must pass through the element B which does not lie in a , it follows that a and b are two distinct general lines lying in Q and must therefore either be parallel to one another, or else have one element in common, which is also an element of P .

Thus a is either parallel to a general line in P or has an element in common with P .

Further, a cannot have more than one element in common with P , since then it would require to lie in P .

THEOREM 154.

If a , b and c be any three distinct general lines having an element O in common, but not all lying in one general plane, and if a general line d , also passing through O , be normal to a , b and c , then d is normal to every general line in the general threefold containing a , b and c .

Let P be the general plane containing b and c .

Then a intersects P in O and so P and a determine a general threefold, say W , containing a , b and c .

Consider now any general line e in W which passes through O but is distinct from a , b and c .

Then a and e determine a general plane, say Q , which by Theorem 149 must lie in W .

Further Q cannot be identical with P , since Q contains a but P does not contain it.

Again Q and P have the element O in common and therefore by Theorem 152 they have a general line, say f , in common which passes through O .

Now since d is normal to the two intersecting general lines b and c , it follows that d is normal to every general line in P and therefore is normal to f .

Again, since d is normal to the two intersecting general lines a and f , it follows that d is normal to every general line in Q and therefore is normal to e .

But e is any general line in W which passes through O but is

distinct from a , b and c and so d is normal to every general line in W which passes through O .

Next let e be any general line in W which does not pass through O and let e' be the general line through O parallel to e .

Then by Theorem 150 e' must also lie in W and so by the first case d is normal to e' and therefore also normal to e .

Thus d is normal to every general line in W as was to be proved.

Definition. A general line which is normal to every general line in a general threefold will be said to be *normal to the general threefold*.

Since, by Theorem 147, if three distinct general lines not all lying in one general plane have an element O in common there is at least one general line passing through O and normal to all three, it follows that through any element of a general threefold there is always at least one general line which is normal to the general threefold.

THE THREE TYPES OF GENERAL THREEFOLD.

As in the case of general lines and general planes there are three types of each, so too there are three types of general threefold.

This may be shown in the following way:

If S be any separation plane and O be any element in it, there is an acceleration plane, say P , which passes through O and is completely normal to S .

Now if a be any general line in P which passes through O , then a must be normal to S and must intersect it.

But a may be either:

- (1) a separation line,
- or (2) an optical line,
- or (3) an inertia line,

and if a general threefold be determined by S and a , then these three cases give rise to the three different types.

Let W be the general threefold determined by a and S and consider first the case where a is a separation line.

If now e be any general line in W which passes through O and is distinct from a , then a and e determine a general plane Q , which lies in W and since Q has the element O in common with S it must have a general line, say f , in common with S .

Now f must pass through O and since it lies in S therefore a must be normal to f .

But a and f are both separation lines and we already know that if

two intersecting separation lines are normal to one another they must lie in a separation plane.

Thus Q must be a separation plane and therefore e must be a separation line.

Thus every general line in W which passes through O must be a separation line.

If e' be any other general line in W which does not pass through O , then there is a general line through O parallel to e' and which by Theorem 150 must also lie in W and therefore must be a separation line.

But a general line parallel to a separation line must itself be a separation line and so e' is a separation line.

Thus every general line in W is a separation line and so no element of W is either *before* or *after* any other element.

It also follows from this that every general plane in W must be a separation plane.

Consider next the case where a is an optical line.

As before let e be any general line in W which passes through O and is distinct from a .

Then a and e determine a general plane Q which has a general line f in common with S .

As before a is normal to f , but in this case a is an optical line while f is a separation line and we know that in these circumstances a and f must lie in an optical plane.

Thus Q must be an optical plane and since there is only one optical line in an optical plane which passes through any element of it and all other general lines in it which pass through that element are separation lines, it follows that e must be a separation line.

Again let e' be any other general line in W which does not pass through O .

Then there is a general line through O parallel to e' and this general line must either be the optical line a or a separation line.

Thus e' must be either an optical line or a separation line.

Again if O' be any element of W distinct from O then O' may or may not lie in a .

If O' does not lie in a then OO' is a separation line and there is an optical line through O' parallel to a and which by Theorem 150 must lie in W .

Thus there is at least one optical line passing through any element of W and lying in W .

Let e' be any general line in W which passes through O' but not through O , and which is not parallel to a .

Then the general line through O parallel to e' cannot be identical with a and therefore must be a separation line.

Thus e' must be a separation line.

It follows that of all the general lines passing through any given element of W and lying in W one and only one is an optical line and all the others are separation lines.

Further all the optical lines in W are parallel to one another.

Since there are two optical lines in any acceleration plane which pass through any element of it, it follows that no acceleration plane can lie in W .

Thus every general plane in W must be either a separation plane or an optical plane.

It follows that all the optical lines in W being parallel to one another must be neutral-parallels.

Consider finally the case where a is an inertia line.

As before let e be any general line in W which passes through O and is distinct from a .

Then a and e determine a general plane Q , which lies in W and, since a is an inertia line, Q must be an acceleration plane.

Thus e may be either an inertia line, an optical line, or a separation line.

If O' be any element in W which is distinct from O and if d be any general line passing through O and lying in W , but distinct from OO' , then through O' there is a general line parallel to d , which must lie in W and must be of the same type as d .

Thus through any element of W there are general lines of all three types lying in W .

Again if f be any general line lying in S and passing through O then, since a is an inertia line, a and f must lie in an acceleration plane, say R .

Now since there are an infinite number of general lines such as f which lie in S and pass through O there must be an infinite number of acceleration planes such as R which are all distinct but have the inertia line a in common.

In any one of these acceleration planes such as R there are two and only two optical lines which pass through O .

All these optical lines must be distinct since the acceleration planes have only an inertia line in common, and so there are an infinite number of optical lines passing through O and lying in W .

Further any optical line which passes through O and lies in W must clearly lie in one of this set of acceleration planes.

Again if O' be any element of W distinct from O and if g be any optical line passing through O and lying in W , but distinct from OO' , then there is an optical line through O' parallel to g and lying in W .

The general line OO' either may or may not itself be an optical line.

Thus through any element of W there are an infinite number of optical lines which lie in W .

Now we have already seen that W contains the separation plane S and also contains acceleration planes and we can easily show that it also contains optical planes.

Thus let P be any acceleration plane in W and let A be any element in W but not in P .

Then through A there is an acceleration plane parallel to P which we may call P' .

Let B be the representative of A in P and let c_1 and c_2 be the two generators of P which pass through B .

Then A is neither *before* nor *after* any element of either c_1 or c_2 and so A and c_1 lie in one optical plane, say T_1 , while A and c_2 lie in another optical plane, say T_2 .

But T_1 and T_2 each contain three elements in common with W which do not all lie in one general line and so, by Theorem 149, both T_1 and T_2 lie in W .

Thus W contains all three types of general plane.

We thus see that there are at least three types of general threefold and we have investigated a few of their characteristic properties.

We have next to show that any general threefold must belong to one of these three types.

Since any four distinct elements which do not all lie in one general plane lie in one and only one general threefold it will be sufficient if we examine the nature of any such general threefold.

SETS OF FOUR ELEMENTS WHICH DETERMINE THE DIFFERENT TYPES OF GENERAL THREEFOLD.

Let A, B, C, D be any four distinct elements which do not all lie in one general plane.

Then no three of them can lie in one general line and A, B and C must determine a general plane which we shall call P .

Now P may be either:

- (1) an acceleration plane,
- or (2) an optical plane,
- or (3) a separation plane.

Suppose first that P is an acceleration plane and that D is any element outside it.

Let W be the general threefold containing A, B, C and D and which must evidently contain P .

Then by Theorem 144 there is one single general line passing through D and normal to P which has an element in common with P .

Let this element be denoted by O and let a be any inertia line in P which passes through O , while b is the separation line in P and passing through O which is normal to a .

Then since a is an inertia line the general line DO which is normal to it must be a separation line.

But DO is also normal to b and since we know that two intersecting separation lines which are normal to one another must lie in a separation plane, it follows that DO and b lie in a separation plane which we shall call S .

Now S contains DO and b and therefore contains three elements in common with W which do not all lie in one general line.

It follows by Theorem 149 that S lies in W .

Thus by Theorem 151 the general plane determined by S and a is identical with the general plane determined by P and DO .

This latter is however identical with W and so S and a determine W .

But a is an inertia line which is normal to the two intersecting separation lines DO and b which lie in S and therefore a is normal to S .

Thus the general threefold W is of the third type.

Further it is evident that if any general threefold contains an acceleration plane it must belong to the third type.

Next consider the case where P is an optical plane and D an element outside it.

Two sub-cases arise here: we may have

D *before* or *after* some element of P ,

or D neither *before* nor *after* any element of P .

We shall suppose first that D is either *before* or *after* some element of P and we shall denote the generator of P which passes through this element by a .

If as before W denote the general threefold containing A, B, C and D , then W will contain P and will therefore contain a .

But since a is an optical line and D is an element which does not lie in a but is either *before* or *after* some element of a , it follows that a and D lie in an acceleration plane, say Q .

But Q contains three elements in common with W which do not all lie in one general line and so Q must lie in W .

But Q is an acceleration plane and so it follows that in this case also W is a general threefold of the third type.

We shall next take the case where P is an optical plane and the element D is neither *before* nor *after* any element of P .

Let b be any separation line in P and a be any optical line in P and let b and a intersect in the element O .

If as before W denote the general threefold containing A, B, C and D , then W will contain P and therefore will contain a and b .

Now, since D is neither *before* nor *after* any element of P , it is neither *before* nor *after* any element of b and so D and b lie in a separation plane which we may call S .

Further, since S has three elements in common with W which do not all lie in one general line, it follows that S lies in W .

Again D and a must lie in an optical plane and, since DO is a separation line while a is an optical line, it follows that a is normal to DO .

But a must also be normal to b for a similar reason and so, since DO and b are intersecting separation lines in S , it follows that a is normal to S .

But by Theorem 151 the general threefold determined by S and a is identical with that determined by P and DO which again is identical with W .

Since however S is a separation plane while a is an optical line normal to it, it follows that W is in this case a general threefold of the second type.

Consider next the case where P is a separation plane and as in the previous cases let W denote the general threefold containing A, B, C and D and therefore also containing P .

Three sub-cases occur here: thus we may have

- D neither *before* nor *after* any element of P ,
- or D either *before* or *after* one single element of P ,
- or D either *before* or *after* at least two elements of P .

Now by Theorem 144 there is one single general line passing through D and normal to P which has an element in common with P .

Let O be this element.

Then DO may be either a separation line, an optical line, or an inertia line.

Consider first the case where D is neither *before* nor *after* any element of P .

Then D is neither *before* nor *after* O and so DO is a separation line and the general threefold W is of the first type.

Next consider the case where D is either *before* or *after* one single element of P and denote this element by O' .

Let b and c be two distinct separation lines in P and passing through O' .

Then DO' and b lie in an optical plane and DO' and c lie in another optical plane.

Since D is either *before* or *after* O' , it follows that DO' is an optical line and therefore is normal to both b and c .

Since b and c intersect one another, it follows that DO' is normal to P and therefore O' must be identical with O .

Thus in this case the general threefold W is of the second type.

Next let D be either *before* or *after* at least two distinct elements of P , say E and F .

Then EF is a separation line and D does not lie in it, and so the three elements D , E and F lie in an acceleration plane, say Q .

But D , E and F are elements in W and therefore Q must lie in W .

Thus since Q is an acceleration plane it follows that the general threefold W belongs in this case to the third type.

This exhausts all the possibilities which are open and so we see that any general threefold whatever must be of one of the three types which we have considered.

We shall accordingly give special names to these three types.

Definition. If a separation line a intersects a separation plane S and is normal to it, then the aggregate of all elements of S and of all separation planes parallel to S which intersect a will be called a *separation threefold*.

Definition. If an optical line a intersects a separation plane S and is normal to it, then the aggregate of all elements of S and of all separation planes parallel to S which intersect a will be called an *optical threefold*.

Definition. If an inertia line a intersects a separation plane S and is normal to it, then the aggregate of all elements of S and of all separation planes parallel to S which intersect a will be called a *rotation threefold*.

We are now in a position to introduce a new postulate which limits the number of dimensions of our set of elements.

POSTULATE XX. **If W be any optical threefold, then any element of the set must be either before or after some element of W .**

If W be any optical threefold and A be any element of W , then through A there is one single optical line which lies in W and A is *before* certain elements of this optical line and is *after* certain others.

Thus in this case A is *before* certain elements of W and *after* certain other elements of W .

If, on the other hand, A be any element outside W , then, by Post. XX, A must be either *before* some element of W or *after* some element of W .

If A be *before* the element B of W , then there is an optical line, say b , passing through B and lying in W .

If b' be the optical line through A parallel to b then b' will be a before-parallel of b .

But any element of W which does not lie in b must lie in an optical line c neutrally parallel to b and lying in W and so by Theorem 25 b' must be a before-parallel of c .

Thus A must be *before* certain elements of c and since A is not an element of W and therefore not an element of c , it follows that A cannot be *after* any element of c .

Thus A is *before* elements of every optical line in W and is not *after* any element of W .

Similarly if A be any element outside W and *after* some element of W , then A will be *after* elements of every optical line in W and will not be *before* any element of W .

Definition. An optical line which lies in an optical or rotation threefold will be spoken of as a *generator* of the optical threefold or rotation threefold, as the case may be.

THEOREM 155.

If P be an optical plane and O be any element in it, there is only one general plane passing through O and completely normal to P .

Let a be the generator of P which passes through O and let b be any separation line in P and passing through O .

Then we already know that there is at least one optical plane, say Q , which passes through O and is completely normal to P .

Further this optical plane Q contains a .

Now let c be any separation line passing through O and lying in Q . Then c is normal to both a and b .

Let d be any other general line which passes through O and is normal to P and let X be any element in d distinct from O .

Now, P and c determine an optical threefold since no element of c with the exception of O is either *before* or *after* any element of P .

Let this optical threefold be denoted by W .

Then, by Post. XX, the element X is either *before* or *after* some element of W .

If X were outside W , then, as we have seen, X would be *before* or *after* elements of every generator of W and therefore *before* or *after* elements of a .

Since the general line d could not then be either identical with a or be a separation line normal to a , it follows that d could not be normal to P , contrary to hypothesis.

Thus X must lie in W and therefore d must lie in W .

But now c and d determine a general plane Q' which has three elements in common with W which are not all in one general line and therefore Q' must lie in W .

Further since P and Q' have the element O in common, therefore by Theorem 152 they have a general line in common, which we may call a' .

But now b is normal to both c and d and, since these intersect in O , it follows that b is normal to Q' and therefore normal to a' .

But b and a' lie in the optical plane P and since b is a separation line a' must be an optical line.

Thus since a' passes through O it must be identical with a and so Q' must be identical with Q .

It follows that d lies in Q and accordingly every general line which passes through O and is normal to P must lie in Q .

Thus any general plane which passes through O and is completely normal to P must be identical with Q , or there is only one general plane passing through O and completely normal to P .

THEOREM 156.

If P be a separation plane and O be any element in it, there is only one general plane passing through O and completely normal to P .

We already know that there is at least one acceleration plane, say Q , passing through O and completely normal to P .

Suppose, if possible, that there is a general line, say a , passing through O and normal to P but not lying in Q .

Then Q and a will determine a rotation threefold, say W .

If b and c be any two distinct general lines in P which both pass through O , then b and c will each be normal to three distinct general lines passing through O and lying in W , but not all lying in one general plane.

Thus, by Theorem 154, b and c must each be normal to every general line in W .

But now we have seen that any rotation threefold contains optical planes and if we take any such optical plane it would either pass through O or else there would be a parallel optical plane passing through O which, by Theorem 150, must also lie in W .

Thus there would always be at least one optical plane, say R , passing through O and lying in W .

But then both b and c would be normal to every general line in R and since b and c are intersecting general lines in P we should have every general line in R normal to every general line in P .

Thus P would be completely normal to R and would pass through the element O in it.

But P is a separation plane and we already know by Theorem 155 that there could be only one general plane passing through O and completely normal to R , and that one must itself be an optical plane and could not be a separation plane.

Thus the assumption that there is a general line a passing through O and normal to P but not lying in Q , leads to a contradiction and therefore is not true.

It follows that every general line passing through O and normal to P must lie in Q .

Thus Q is the only general plane which passes through O and is completely normal to P .

Thus the theorem is proved.

THEOREM 157.

If P be an acceleration plane and O be any element in it, there is only one general plane passing through O and completely normal to P .

We already know that there is at least one separation plane, say Q , passing through O and completely normal to P .

Let b be any separation line in Q which passes through O and let c be the one separation line lying in Q and passing through O which is normal to b .

Suppose now, if possible, that there is a general line d passing through O and normal to P but not lying in Q .

Then, since any inertia line in P would be normal to d , it would

follow that d must be a separation line and since then any inertia line in P which passed through O would be conjugate to the two intersecting separation lines b and d it would follow, as a consequence of Theorem 99, that b and d must lie in a separation plane, say R .

Now R would require to be distinct from Q , since d is supposed not to lie in Q .

Since however we should then have two intersecting separation lines in R : namely b and d , normal to P , it would follow that R was completely normal to P .

Now suppose e to be the one separation line in R and passing through O which would be normal to b .

Then c and e would be distinct separation lines since b is the only general line common to Q and R .

Further since any inertia line in P would be normal to both c and e it follows that c and e would lie in a separation plane, say S .

But now P and b would determine a rotation threefold, say W , and since both c and e would be normal to P and to the separation line b (which does not lie in P) it follows, by Theorem 154, that both c and e would be normal to every general line in W .

But, as we have seen, there is at least one optical plane passing through O and lying in W and if T be such an optical plane we should have both c and e normal to T .

Thus the separation plane S would be completely normal to T and this we know by Theorem 155 is impossible, since only an optical plane can have an element in common with an optical plane and be completely normal to it.

It follows that no such general line as d can exist and so every general line which passes through O and is normal to P must lie in Q .

Thus Q is the only general plane which passes through O and is completely normal to P and so the theorem is proved.

REMARKS.

Combining these last three theorems we get the general result:

If P be any general plane and O be any element in it, there is one and only one general plane Q passing through O and completely normal to P .

Further:

If P be an optical plane, Q is an optical plane.

If P be a separation plane, Q is an acceleration plane.

If P be an acceleration plane, Q is a separation plane.

Again we know that if O' be any element outside P there is at least one general plane through O' which is completely normal to P .

If we call this general plane Q' , then Q' is either identical with Q or parallel to Q according as O' does or does not lie in Q .

Now there cannot be any other general plane than Q' which passes through O' and is completely normal to P .

For if Q'' were such another general plane it would either pass through O or else there would be a general plane parallel to Q'' and passing through O , which would also be completely normal to P .

Thus there would be two distinct general planes passing through O and completely normal to P ; which is impossible.

Thus we can say:

If P be any general plane and O be any element of the set, there is one and only one general plane passing through O and completely normal to P .

THEOREM 158.

(1) *If P be an acceleration or separation plane and O be any element outside it, then the general plane through O and completely normal to P has one single element in common with P .*

(2) *If P be an optical plane and O be any element outside it, then the optical plane through O and completely normal to P has an optical line in common with P if O be neither before nor after any element of P and has no element in common with P if O be either before or after any element of P .*

Let P be an acceleration or separation plane and O any element outside it.

Then by Theorem 144 there is one single general line passing through O and normal to P which has an element in common with P .

Let O' be this element.

Then by Theorem 157 or 156 there is one single separation or acceleration plane, say Q , which passes through O' and is completely normal to P ; and Q has only one element in common with P .

Thus Q must contain the general line $O'O$ and therefore it must be identical with the one single general plane which passes through O and is completely normal to P .

Thus the general plane through O and completely normal to P has one single element in common with P , and so the first part of the theorem is proved.

Next let P be an optical plane and O any element outside it.

Then, by Theorem 145, if O be neither *before* nor *after* any element

of P there is one single generator of P such that every general line which passes through O and intersects this generator is normal to P .

Thus if a be this generator and O' be any element in a , the general lines a and OO' determine an optical plane, say Q , which passes through O , is completely normal to P and has the optical line a in common with P .

Since there is only one optical plane through O and completely normal to P , this must be identical with Q and it has the optical line a in common with P if O be neither *before* nor *after* any element of P .

Next consider the case where O is either *before* or *after* some element of P .

Here, by Theorem 145, there is no general line passing through O and having an element in common with P which is normal to P .

Thus the optical plane through O and completely normal to P can, in this case, have no element in common with P .

Thus all parts of the theorem are proved.

THEOREM 159.

If a general line a have an element O in common with a general threefold W , then there is at least one general plane lying in W and passing through O to which a is normal.

Let Q be any general plane in W and passing through O .

Then by Theorem 146 there is at least one general line, say b , passing through O and lying in Q which is normal to a .

Let c be any other general line distinct from b , lying in Q and passing through O and let A be any element lying in W but not in Q .

Then c and A determine a general plane, say R , which must lie in W since it contains three elements in common with W which do not all lie in one general line.

Further R must be distinct from Q , since R contains the element A which does not lie in Q , and moreover R does not contain b .

But again, by Theorem 146, there is at least one general line, say d , passing through O and lying in R which is normal to a .

Then d must be distinct from b which it intersects in the element O and so d and b determine a general plane, say P , which must lie in W since it contains three elements in common with W which do not all lie in one general line.

But since a is normal to the two intersecting general lines d and b , therefore a is normal to P , and thus there is at least one general plane P lying in W and passing through O to which a is normal.

It is to be observed in connection with the above theorem that if a were normal to any other general line passing through O and lying in W but not in P , then, by Theorem 154, a would be normal to every general line in W .

It is also to be observed that the above theorem holds both when the general line a lies in W and when it has only one element in common with W .

THEOREM 160.

(1) *If W be a general threefold and P be a general plane lying in W , while O is any element in P , then there is at least one general line passing through O and lying in W which is normal to P .*

(2) *There is only one such general line except in the case where W is an optical threefold and P an optical plane, in which case there are an infinite number.*

To prove the first part of the theorem consider first the case where P is an optical plane.

In this case the generator of P which passes through O is normal to P and lies in W .

Next let P be an acceleration or separation plane and let A be any element lying in W but not in P .

Then by Theorem 144 there is one single general line passing through A and normal to P which has an element in common with P .

Let B be this element.

Then the general line AB has two distinct elements in common with W and therefore lies in W , but does not lie in P .

If B should be identical with O , then AB passes through O , lies in W and is normal to P .

If B be not identical with O , then there is a general line passing through O and parallel to AB which must also be normal to P .

But by Theorem 150 this general line must also lie in W .

Thus in all cases there is at least one general line passing through O and lying in W which is normal to P .

Proceeding now to the second part of the theorem, let us consider first the case where P is either an acceleration or separation plane.

Suppose, if possible, that a and b are two distinct general lines both of which pass through O , lie in W and are normal to P .

Then a and b would determine a general plane, say Q , which would have three elements in common with W not all lying in one general line, and so Q would lie in W .

Thus by Theorem 152, since Q and P have the element O in common they would have a general line in common.

But, since Q is supposed to contain the two intersecting general lines a and b each of which is normal to P , it would follow that Q must be completely normal to P , and since P is by hypothesis either an acceleration or separation plane, it would follow that Q must be either a separation or acceleration plane.

But we already know that if an acceleration plane and a separation plane be completely normal to one another, they cannot have more than one element in common.

Thus P and Q could not have a general line in common and so the supposition that more than one general line can pass through O , lie in W , and be normal to P leads in this case to a contradiction and therefore is not true.

Thus if P be an acceleration or separation plane there cannot be more than one such general line.

Suppose next that P is an optical plane and let a be the generator of P which passes through O and let b be any general line lying in W but not in P and which passes through O .

Let A be any element in b distinct from O .

Then if A be either *before* or *after* any element of P the general threefold W must be a rotation threefold and a and A must lie in an acceleration plane.

Thus since a is an optical line and since b intersects a and lies in an acceleration plane with it, it follows that b cannot be normal to a and therefore cannot be normal to P .

Further, since a is the only general line in P which passes through O and is normal to P , it follows that in this case there is only one general line in W which passes through O and is normal to P .

Consider now the case where the element A is neither *before* nor *after* any element of P .

In this case the general threefold W must be an optical threefold and the general line b must be a separation line.

Let c be any general line in P and passing through O but distinct from a .

Then c is a separation line and b and c determine a separation plane, say S , which must lie in W .

Now a must, in this case, be normal to both b and c and therefore normal to S .

Let d be the one single separation line in S which passes through O and is normal to c .

Then d is normal to both a and c and therefore is normal to P .

If then Q be the general plane containing a and d , it contains two intersecting general lines each of which is normal to P and therefore it follows that Q is completely normal to P .

Thus every general line which passes through O and lies in Q must be normal to P .

But since a and d are two intersecting general lines which both lie in W , it follows that Q contains three distinct elements in common with W which do not all lie in one general line and therefore Q must lie in W .

Thus in this case there are an infinite number of general lines which pass through O , lie in W , and are normal to P .

This exhausts all the different cases and so the second part of the theorem is proved.

THEOREM 161.

If W be a general threefold and O be any element which does not lie in it, then :

(1) *If W be a rotation or separation threefold there is one single general line passing through O and normal to W which has an element in common with W .*

(2) *If W be an optical threefold there is no general line passing through O and normal to W which has an element in common with W .*

If W be a rotation threefold it contains inertia lines.

Let f be any inertia line in W .

Then f and O lie in an acceleration plane, say R , and if a be any inertia line in R and passing through O but not parallel to f then a and f will intersect in some element, say A , which is an element of W .

If on the other hand W be a separation threefold, let A be any element in W and let a be the general line OA .

Now whether W be a rotation or separation threefold, it follows by Theorem 159 that there is at least one general plane, say P , lying in W and passing through A to which a is normal.

Now if W be a rotation threefold, a has been selected so as to be an inertia line and, since only separation lines can be normal to an inertia line, it follows that P is a separation plane.

If on the other hand W be a separation threefold it can contain no other type of general plane, and so in this case also P must be a separation plane.

Now by Theorem 160, whether W be a rotation or a separation threefold there is one general line, say b , passing through A and lying

in W which is normal to P , and, since P is a separation plane, b must intersect it.

Now a and b must be distinct, since b lies in W while a can only have the one element A in common with W .

Thus a and b lie in a general plane, say Q , and, since Q contains two intersecting general lines each of which is normal to P , it follows that Q must be completely normal to P .

Further, since P is a separation plane, it follows that Q is an acceleration plane.

Now since b is normal to P and lies in W , the general threefold W might be determined by P and b and we know that if b be a separation line, W must be a separation threefold, while if b be an optical line, W must be an optical threefold and if b be an inertia line W must be a rotation threefold.

It follows that if W be a rotation threefold then b must be an inertia line, while if W be a separation threefold, b must be a separation line.

But now in either of these cases there is a general line, say c , which passes through O , lies in Q and is normal to b , and in both cases c intersects b in some element, say O' , which is an element of W .

Further c will be a separation line if b be an inertia line: that is, if W be a rotation threefold; while c will be an inertia line if b be a separation line: that is, if W be a separation threefold.

Now since c lies in Q and since Q is completely normal to P , it follows that c is normal to P .

If then P' be a general plane passing through O' and parallel to P it follows by Theorem 150 that P' must also lie in W .

Thus c will be normal to P' and to the general line b which intersects P' in O' .

It is thus evident that c is normal to three distinct general lines in W which have the element O' in common and which do not all lie in one general plane and therefore, by Theorem 154, c is normal to W .

Also c passes through O and has the element O' in common with W .

Now there can be no other general line passing through O and normal to W ; for suppose, if possible, that c' is such another general line.

Then c and c' would determine a general plane, say T , which would contain two intersecting general lines each of which would be normal to every general line in W and therefore normal to every general plane in W .

Thus T would be completely normal to every general plane in W .

But through any element of W there passes more than one general plane which lies in W and so we should have more than one general plane passing through any element of W and completely normal to T , which, as we have seen, is impossible.

Thus the supposition that more than one general line can pass through O and be normal to W leads to a contradiction and therefore is not true.

Thus there is one and only one general line which passes through O and is normal to W when W is a rotation or separation threefold, and this general line has an element in common with W .

Suppose next that W is an optical threefold.

Then, by Post. XX, O must be either *before* or *after* some element of W and, as we have seen, if O be *before* any element of W it must be *before* elements of every generator of W , while if O be *after* any element of W it must be *after* elements of every generator of W .

If then a be any general line which passes through O and has an element A in common with W , then A must lie in some generator of W , say f , and f and a will lie in an acceleration plane.

But since f is an optical line and a is a general line intersecting f and lying in an acceleration plane with it, it follows that a cannot be normal to f and therefore cannot be normal to W .

Thus in this case there is no general line passing through O and normal to W which has an element in common with W .

Thus both parts of the theorem are proved.

REMARKS.

If W be a rotation or separation threefold and O be any element in W , it is easy to see that there is one and only one general line passing through O and normal to W .

For if A be any element outside W and a be the one general line passing through A and normal to W , then a will have an element B in common with W .

If B should coincide with O , then a is a general line passing through O and normal to W .

If B does not coincide with O , then a general line a' passing through O and parallel to a must be normal to every general line in W and must therefore be normal to W .

Thus we have shown that there is at least one general line passing through O and normal to W and the same considerations employed in the last theorem show that there is only one such general line.

Further the general line through O normal to W cannot have more

than the one element O in common with W ; for if it had a second element in common with W it would lie entirely in W , and by Theorem 150 it would follow that a must lie in W , contrary to the hypothesis that the element A of a lies outside W .

In this respect an optical threefold is quite different.

Through any element O in an optical threefold W there passes one single generator of W , say a .

Now a is normal to any separation line in W and is also normal to itself.

Thus a is normal to W and passes through O , but lies entirely in W .

If O' be any element outside W and a' be an optical line parallel to a , then a' is also normal to W but can have no element in common with W .

We may also show, by similar considerations to those employed in the case of a rotation or separation threefold, that there cannot be more than one general line passing through any element and normal to a given optical threefold.

Thus for all three types of general threefold we have the result:

If W be any general threefold and O be any element of the set, there is one and only one general line passing through O and normal to W .

THEOREM 162.

If a be a general line and O be any element in it, there is one and only one general threefold passing through O and normal to a .

Let P be any acceleration plane containing a and let Q be the separation plane passing through O and completely normal to P .

Then P and Q have only the one element O in common.

Now through O and lying in P there is one single general line, say b , which is normal to a .

But b and Q can have only one element in common and therefore they determine a general threefold, say W .

Since however a is normal to every general line in Q and is also normal to the general line b which passes through O and does not lie in Q , it follows by Theorem 154 that a is normal to W .

Thus there is at least one general threefold passing through O and normal to a .

We shall next show that every general line which passes through O and is normal to a must lie in W .

Since every such general line which lies in Q must lie in W , it will be sufficient to consider any general line c passing through O , normal to a and not lying in Q .

Then c and Q determine a general threefold, say W' , and by Theorem 160 there is at least one general line, say d , passing through O and lying in W' which is normal to Q .

Further, since Q is a separation plane, d must lie in the acceleration plane through O which is completely normal to Q , and since there is only one such acceleration plane, it follows that d must lie in P .

But since a is normal to c and Q it follows that a is normal to W' and therefore is normal to d .

But there is only one general line passing through O and lying in P which is normal to a , and by hypothesis b is this general line.

It follows that d must be identical with b and so, by Theorem 151, since d and Q must determine the same general threefold as do c and Q , it follows that W' must be identical with W .

Thus c must lie in W .

But if there were any other general threefold distinct from W which passed through O and was normal to a , such general threefold would require to contain a general line which passed through O and was normal to a , but which did not lie in W and this we have shown to be impossible.

Thus there is one and only one general threefold which passes through O and is normal to a .

REMARKS.

In the above theorem it is to be observed that: if a be an inertia line, b must be a separation line; if a be a separation line, b must be an inertia line; while if a be an optical line, b must be the same optical line.

Thus it follows that: if a be a general line and O be any element in it, while W is a general threefold passing through O and normal to a , then:

- (1) If a be an inertia line, W is a separation threefold.
- (2) If a be a separation line, W is a rotation threefold.
- (3) If a be an optical line, W is an optical threefold containing a .

On the other hand we have already seen that if W be a general threefold and O be any element in it, there is one and only one general line a passing through O and normal to W .

Thus it follows that:

- (1) If W be a separation threefold, a is an inertia line.
- (2) If W be a rotation threefold, a is a separation line.
- (3) If W be an optical threefold, a is an optical line lying in W .

Again if a be a general line and O be any element which does not lie in a , then, through O there is one single general line, say a' , which is parallel to a and is accordingly a general line of the same type.

Thus through O there is a general threefold which is normal to a' and therefore also normal to a .

Further there cannot be a second general threefold passing through O and normal to a , for such general threefold would also be normal to a' and so we should have two general threefolds passing through O and normal to a' contrary to Theorem 162.

Thus we can extend Theorem 162 and say:

If a be a general line and O be any element of the set, there is one and only one general threefold passing through O and normal to a .

THEOREM 163.

If W be an optical threefold and A be any element outside it, then every optical line through A , except the one parallel to the generators of W , has one single element in common with W .

Let a be the optical line through A parallel to the generators of W and let b be any such generator.

Then, by Post. XX, A must be either *before* or *after* some element of W and we have already seen that if A be *before* an element of W it must be *before* elements of every generator of W ; while if A be *after* an element of W it must be *after* elements of every generator of W .

Thus a must be either a before- or after-parallel of b .

It will be sufficient to consider the case where a is a before-parallel of b since the proof in the other case is quite analogous.

Then a and b lie in an acceleration plane and so there is one single optical line passing through A and intersecting b in some element, say B .

If we call this optical line c , then c has the element B in common with W .

If then d be any optical line passing through A but distinct from c and a , it follows, by Post. XII, that there is one single element in d , say D , which is neither *before* nor *after* any element of b .

Now if D were outside W it would be either *before* or *after* elements of every generator of W , as we have already seen.

Thus, since D is neither *before* nor *after* any element of the generator b , it follows that D must lie in W .

It follows that every optical line through A with the exception of a has at least one element in common with W .

But if any optical line has more than one element in common with W it must lie entirely in W , which is not possible for any optical line which passes through the element A .

It follows that every optical line through A with the exception of a has one single element in common with W , as was to be proved.

THEOREM 164.

If W be a general threefold and A be any element outside it, then any general line through A is either parallel to a general line in W or else has one single element in common with W .

It will be observed that the last theorem is a special case of this one.

Let a be any general line which passes through A .

Now a cannot have more than one element in common with W , for then it would require to lie entirely in W and therefore could not pass through A .

Let B be a second element in a distinct from A .

In case W be a rotation or separation threefold, let the general line through A normal to W meet W in the element A' , as we have seen in Theorem 161 that it must.

Now in case the general line a should coincide with AA' it would have an element in common with W , and so we shall suppose it is distinct from it.

Again let the general line through B normal to W meet W in the element B' .

Then since B does not lie in AA' we must have BB' parallel to AA' .

In case W be an optical threefold then, by Theorem 163, any optical line through A except the one parallel to the generators of W must have an element in common with W .

Let any optical line through A which is not parallel to the generators of W meet W in the element A' .

In case the general line a should coincide with AA' it would have an element in common with W , and so we shall suppose it is distinct from it.

Let the optical line through B parallel to AA' meet W in the element B' .

Now both in the cases where W is a rotation or separation threefold and where W is an optical threefold, since BB' is parallel to AA' , it follows that BB' and AA' lie in a general plane which we may call Q .

But $A'B'$ and a must also lie in Q , and therefore a is either parallel to $A'B'$ or intersects $A'B'$ in some element, say C .

But $A'B'$ has two distinct elements A' and B' in common with W ,

and therefore $A'B'$ must lie in W , and if the element C exists it must lie in W .

Thus the general line a is either parallel to a general line in W or else a has one single element in common with W .

Definition. If a general line and a general threefold have one single element in common, they will be said to *intersect* in that element.

REMARKS.

Since a separation threefold contains neither an inertia nor an optical line it is evident that it can contain no general line which is parallel to either of these.

Thus it follows from the last theorem that: *every inertia and every optical line intersects every separation threefold.*

Again, an optical threefold does not contain any inertia line, and all the optical lines which it contains are parallel to one another.

Thus: *every inertia line and every optical line which is not parallel to a generator of an optical threefold intersects the optical threefold.*

Analogous results to these may be deduced from Theorem 153, with regard to the intersection of certain types of general lines with certain types of general planes.

Thus, since a separation plane contains neither an inertia nor an optical line, it follows from Theorem 153 that: *if W be a rotation threefold every inertia and every optical line in W intersects every separation plane in W .*

Similarly: *if W be a rotation threefold, every inertia line in W and every optical line in W which is not parallel to a generator of an optical plane in W intersects the optical plane.*

Again: *if W be an optical threefold every optical line in W intersects every separation plane in W .*

THEOREM 165.

If W be a general threefold and P be a general plane which does not lie in W , then if P has one element in common with W , it has a general line in common with W .

Let P and W have the element A in common and let B be any element in P which does not lie in W .

Let b be any general line in P which passes through B but is distinct from BA .

Then, by Theorem 164, b must either intersect W in some element, say C , or else b must be parallel to some general line, say b' , which lies in W .

In the first case P and W have the two distinct elements A and C in common and therefore have the general line AC in common.

In the second case a general line b'' passing through A and parallel to b' or identical with it must lie in W .

But b'' must be parallel to b and since it passes through the element A of P it must lie in P .

Thus in this case P and W have the general line b'' in common and so the theorem holds in general.

THEOREM 166.

If W_1 and W_2 be two distinct general threefolds having an element A in common, then they have a general plane in common.

Let B be any element which lies in W_1 but not in W_2 .

Then the general line AB lies in W_1 .

Let Q and R be any two distinct general planes which contain the general line AB and which lie in W_1 .

Then Q does not lie in W_2 but has the element A in common with W_2 and therefore, by Theorem 165, Q has a general line, say a , in common with W_2 .

Similarly R has a general line, say b , in common with W_2 .

Now both a and b must be distinct from the general line AB since the latter does not lie in W_2 and, since Q and R have only the general line AB in common, it follows that b is distinct from a .

Thus a and b are two general lines intersecting in A and each of them lying both in W_1 and W_2 and so they determine a general plane, say P .

But P contains three elements in common both with W_1 and W_2 and which do not all lie in one general line and so P lies both in W_1 and W_2 .

Thus W_1 and W_2 have a general plane in common.

THEOREM 167.

If P_1 and P_2 be two general planes having no element in common, then through any element of either of them there is at least one general line lying in that general plane which is parallel to a general line in the other general plane.

Let O_1 be any element in P_1 and let O_2 be any element in P_2 and let the general line O_1O_2 be denoted by a .

Then P_1 and a determine a general threefold, say W_1 , while P_2 and a determine a general threefold, say W_2 .

If W_2 should be identical with W_1 , then P_1 and P_2 lie in one general threefold and, since they have no element in common, it follows by Theorem 153 that any general line in P_1 is parallel to a general line in P_2 , and so P_1 and P_2 are parallel to one another.

If W_2 be not identical with W_1 then, since W_1 and W_2 have all the elements of a in common, it follows, by Theorem 166, that they have a general plane, say Q , in common which must contain a .

But now Q must be distinct from both P_1 and P_2 , for otherwise P_1 or P_2 would contain a and so P_1 and P_2 would have an element in common, contrary to hypothesis.

But now P_1 and Q both lie in W_1 and they have the element O_1 in common, and therefore, by Theorem 152, they have a general line, say b_1 , in common, which passes through O_1 .

Similarly P_2 and Q have a general line, say b_2 , in common, which passes through O_2 .

But since b_1 and b_2 lie in P_1 and P_2 respectively they can have no element in common, and, since they both lie in the general plane Q , they must be parallel to one another.

Thus the theorem is proved.

REMARKS.

It is easy to see that if two general planes P_1 and P_2 have one single element O in common, then no general line in P_1 can be parallel to any general line in P_2 .

For let a_1 and a_2 be two general lines in P_1 and P_2 respectively, then a_1 cannot be parallel to a_2 if both pass through O .

Further, they cannot be parallel if one passes through O and the other does not, for then they could not lie in one general plane.

Finally they cannot be parallel if neither of them passes through O , for then a general line a_1' passing through O and parallel to a_1 would lie in P_1 and so could not be parallel to a_2 as it would require to be if a_2 were parallel to a_1 .

THEOREM 168.

If W be a general threefold and O be any element outside it, and, if further, a and b be two distinct general lines intersecting in O and each of them parallel to a general line in W , then:

(1) *The general plane containing a and b has no element in common with W .*

(2) *The general plane containing a and b is parallel to a general plane in W .*

Neither a nor b can have any element in common with W , since, by Theorem 150, if either of them had an element in common with W , it would require to lie entirely in W and so could not contain the element O .

But, if P be the general plane containing a and b , any element in P must lie either in a or in a general line parallel to a and intersecting b .

But every general line of this character must be parallel to the general line in W to which a is parallel, and therefore can have no element in common with W .

Thus P can have no element in common with W .

In order to prove the second part of the theorem, let a' and b' be general lines in W to which a and b are respectively parallel.

Then a' and b' either intersect, in which case they lie in a general plane which lies in W and is parallel to P , or else a general line, say b'' , parallel to b' , may be taken through any element of a' and then b'' must lie in W , by Theorem 150.

Thus in this case a' and b'' will lie in a general plane which will lie in W and be parallel to P .

Thus in all cases P will be parallel to a general plane in W .

Definition. If W be a general threefold and if through any element A outside W a general line a be taken parallel to any general line in W , then the general line a will be said to be *parallel* to the general threefold W .

Definition. If W be a general threefold and if through any element A outside W a general plane P be taken parallel to any general plane in W , then the general plane P will be said to be *parallel* to the general threefold W .

THEOREM 169.

If W be a general threefold and O be any element outside it, and if through O there pass three general lines a , b , and c , which do not all lie in one general plane and which are respectively parallel to three general lines in W , then a , b and c determine a general threefold W' , such that every general line in W' is parallel to a general line in W .

Let P be the general plane containing b and c .

Then since a , b and c do not lie in one general plane it follows that a can only have the one element O in common with P .

Now the general line a can have no element in common with W , for then, since it is parallel to a general line in W , it would, by Theorem 150, require to lie in W and so could not contain the element O .

Again, by Theorem 168, the general plane P can contain no element in common with W , nor can any general plane which is parallel to P and which intersects a .

But now any element in W' must either lie in P or in a general plane parallel to P and intersecting a .

Thus no element in W' can lie in W , and so no general line in W' can have an element in common with W .

Thus, by Theorem 164, any such general line must be parallel to a general line in W .

Similarly any general line in W must be parallel to a general line in W' .

Definition. If W be a general threefold and if through any element A outside W three general lines be taken not all lying in one general plane but respectively parallel to three general lines in W , then the three general lines through A determine a general threefold which will be said to be *parallel* to W .

REMARKS.

Since a general line can only be parallel to a general line of the same kind, and since if one general threefold be parallel to another, any general line in either of them is parallel to a general line in the other, it follows that a general threefold can only be parallel to a general threefold of the same kind.

Again if W be a general threefold and A be any element outside it, while W' is a general threefold through A parallel to W , then since W' contains the general line through A parallel to any general line in W , the general threefold W' must be uniquely determined when we know W and A .

Also, since two distinct general lines which are parallel to a third general line are parallel to one another, it follows that: *two distinct general threefolds which are parallel to a third general threefold are parallel to one another.*

Again, from Theorem 164, it is evident that: *if W be a general threefold and A be any element outside it, then any general line through A must either lie in the general threefold passing through A and parallel to W , or else must intersect W .*

It is also to be noted that if a general threefold W be normal to a general line a , then any general threefold W' parallel to W must also be normal to a .

OTHER CASES OF NORMALITY.

We have already considered the normality of a general line to a general line, a general plane, or a general threefold.

We have also considered the complete normality of a general plane to a general plane.

These are the only cases in our geometry in which the normality of n -folds is complete.

Thus it is not possible to have every general line in a general plane P normal to every general line in a general threefold W , for then we should have more than one general plane passing through any element of W and completely normal to P , which, as we have seen, is impossible.

For a similar reason we cannot have every general line in a general threefold W_1 normal to every general line in a general threefold W_2 .

The most we can have in these directions is to have a general plane P through any element of which there is one single general line lying in P which is normal to a general threefold W ; or to have a general threefold W_1 through any element of which there is one single general line lying in W_1 which is normal to a general threefold W_2 .

Again, we may have a general plane P_1 through any element of which there is one single general line lying in P_1 which is normal to a general plane P_2 .

In these cases we have what may be described as partial normality.

In ordinary three dimensional geometry the normality of two planes is of this partial character.

Since it is desirable, so far as is possible, to have our nomenclature in conformity with that employed in ordinary geometry, we shall find it convenient to describe the general planes and general threefolds in the above cases as *normal* to one another.

Thus we may have general planes *normal* to one another or *completely normal* to one another: the expression 'normal' by itself being taken to mean partially normal.

In the case of a general plane or a general threefold which is partially normal to a general threefold the word *normal* may be used by itself without any ambiguity.

Thus we have the following definitions:

Definition. A general plane P_1 will be said to be *normal* to a general plane P_2 if through any element of P_1 there is one single general line lying in P_1 which is normal to P_2 .

Definition. A general plane P will be said to be *normal* to a general threefold W if through any element of P there is one single general line lying in P which is normal to W .

Definition. A general threefold W_1 will be said to be *normal* to

a general threefold W_2 if through any element of W_1 there is one single general line lying in W_1 which is normal to W_2 .

It is evident in the above three definitions we might substitute the word *every* for the word *any*.

It is easy to see that if a general plane P_1 be normal to a general plane P_2 , then P_2 will be normal to P_1 .

It will be sufficient to consider the case where P_1 and P_2 have an element A in common.

Let a be the one single general line lying in P_1 and passing through A which is normal to P_2 and let b be any other general line in P_1 which passes through A .

Then by Theorem 146 there is at least one general line, say c , passing through A and lying in P_2 which is normal to b .

But c must also be normal to a and therefore c must be normal to P_1 .

Further, there cannot be more than one general line passing through A and lying in P_2 which is normal to b , unless P_1 be completely normal and not merely normal to P_2 .

Again, a separation line a may be normal to all three types of general plane and also may lie in all three types of general plane.

If then a be normal to any general plane P_1 and if P_2 be any general plane containing a but not completely normal to P_1 then P_2 will be normal to P_1 .

Thus any type of general plane may be normal to any type of general plane.

In particular, since an optical plane contains a series of optical lines which are normal to it, it follows that an optical plane is normal to itself.

It is evident from the definitions that, if a general plane P be normal to a general threefold W , then P will be either simply normal or completely normal to any general plane in W .

THEOREM 170.

If a general plane P be normal to a general threefold W , then through any element of W there is one single general plane lying in W and completely normal to P .

It will be sufficient if we prove the existence of one general plane, say Q , which lies in W and is completely normal to P .

For let O' be any element in W but not in Q and let Q' be a general plane through O' parallel to Q .

Then, by Theorem 150, if Q lies in W , Q' must also lie in W , and if Q be completely normal to P then Q' must also be completely normal to P .

Further, we know that there cannot be more than one general plane passing through a given element and completely normal to P .

We shall therefore proceed to show that a general plane such as Q exists.

Since P is normal to W , there is one single general line passing through any element of P and lying in P which is normal to W .

Let a be any such general line.

Then, if W be a rotation or separation threefold, a will intersect W in some element, say O , while if W be an optical threefold, a will be an optical line either entirely in W , or entirely outside W .

We shall first consider the case where W is a rotation or separation threefold.

In this case a cannot lie in W , but has the element O in common with it.

It follows, by Theorem 165, that P and W have a general line in common which we shall call b and which must be distinct from a .

But now, by Theorem 159, there is a general plane, say Q , passing through O and lying in W to which b is normal.

Since however a is normal to W , it follows that a also is normal to Q .

Thus we have the two intersecting general lines a and b both lying in P and both normal to every general line in Q .

It follows that Q is completely normal to P .

Consider next the case where W is an optical threefold.

Here a must be either a generator of W , or else be parallel to a generator and lie completely outside W .

Suppose first that a is completely outside W .

If we take any general line in P which intersects a , then this general line either intersects W , or does not intersect W .

If it does not intersect W , then, by Theorem 168, P must be parallel to some general plane, say P' , lying in W .

Now, since P contains an optical line, P' must contain an optical line, and since P' lies in the optical threefold W , it follows that P' (and therefore P) is an optical plane.

Let a' be any generator of P' and let O be any element in a' .

Then, by Theorem 162, there is one and only one general threefold passing through O and normal to a' and this must be identical with W .

Thus every general line which passes through O and is normal to a' must lie in W .

But through O there passes one single optical plane, say Q , which is completely normal to P' , and, since therefore a' must be normal to every general line in Q , it follows that Q must lie in W .

Then, since P is parallel to P' , it follows that Q is completely normal to P .

Next suppose that a lies outside W , as in the last case, but that P is not parallel to any general plane in W .

In this case any general line in P which intersects a must intersect W .

Thus P has an element, and therefore must have a general line, say a' , in common with W .

Further, since a has no element in common with W , it follows that a' must be parallel to a and therefore must be an optical line.

Now let O be any element in a' and let b be any general line passing through O and lying in P , but distinct from a' .

Then, by Theorem 159, there is at least one general plane, say Q , lying in W and passing through O to which b is normal.

But a' is normal to W and therefore normal to Q , and so we have the two intersecting general lines a' and b both normal to Q and both lying in P .

Thus Q must be completely normal to P .

Next consider the cases where a is a generator of W .

Here P may either lie in W or not lie in W .

Suppose first that P lies in W .

Then P contains an optical line and therefore is an optical plane, and, as we have already proved in connection with the second case, there is an optical plane, say Q , passing through any element of P and lying in W which is completely normal to P .

Finally let us take the case where a is a generator of W , but where P does not lie in W .

Here we have only to notice that if we take any optical line a' lying in P and parallel to a , then a' must lie entirely outside W , and this case becomes identical with the third one considered.

Thus in all cases a general plane, such as Q , exists, and so the theorem is proved.

THEOREM 171.

If a general threefold W_1 contain a general line which is normal to a general threefold W_2 , then W_2 contains a general line which is normal to W_1 .

Let a be a general line lying in W_1 and which is normal to W_2 .

Suppose first that W_2 is a rotation or separation threefold; then a will be a separation or inertia line which intersects W_2 in some element, say O .

Then W_1 has the element O in common with W_2 , and so, by Theorem 166, W_1 and W_2 have a general plane, say P , in common.

Now, by Theorem 160, there is one single general line, say b , passing through O and lying in W_2 which is normal to P .

But, since a is normal to W_2 , it follows that b is normal to a .

Further, since a is a separation or inertia line it cannot lie in the general plane P , to which it is normal.

Thus b will be normal to three general lines in W_1 all passing through O and which do not all lie in one general plane.

Thus b must be normal to W_1 .

Next suppose that W_2 is an optical threefold.

Then a must be an optical line and must either lie entirely in W_2 or else entirely outside W_2 , but parallel to a generator of W_2 .

If a should lie in W_2 then clearly we might have W_1 identical with W_2 .

If however a lies in W_2 but if W_1 be not identical with W_2 , then there will be elements of W_1 which do not lie in W_2 , and through any such element there would be a general line, say a' , parallel to a and lying entirely outside W_2 but inside W_1 .

Since a' would also be normal to W_2 , it follows that this case may always be treated as a case in which there is a general line lying in W_1 which is normal to W_2 but entirely outside W_2 .

We shall therefore consider the case where a is an optical line lying entirely outside W_2 .

In this case we may have W_1 either parallel to W_2 or not parallel to W_2 .

If W_1 be parallel to W_2 then W_1 must be an optical threefold and a must be normal to it.

Thus any generator b of W_2 will be parallel to a and therefore normal to W_1 .

Thus, since b lies in W_2 , the theorem holds also in this case.

Next suppose a lies entirely outside W_2 , but that W_1 is not parallel to W_2 .

Then there must be some general line in W_1 which is not parallel to any general line in W_2 and which must therefore intersect W_2 .

Thus W_1 and W_2 have an element, and therefore, by Theorem 166, have a general plane, say P , in common.

Since a has no element in common with W_2 , it can have no element

in common with P , and, since a and P both lie in W_1 , it follows, by Theorem 153, that a is parallel to a general line in P .

Thus, since a is an optical line, P must contain an optical line, and, since P lies in W_2 , it follows that P must be an optical plane.

Now let a' be any generator of P and let O be any element in a' .

Then we saw in the course of proving the last theorem that the one single optical plane which passes through O and is completely normal to P must lie in the optical threefold W_2 .

Let Q be this optical plane and let c be any general line which passes through O and lies in W_1 but not in P .

Then, by Theorem 146, there is at least one general line, say b , passing through O and lying in Q which is normal to c .

But since b lies in Q it is normal to every general line in P .

Thus b is normal to three general lines all passing through O and lying in W_1 but not all lying in one general plane.

It follows that b is normal to W_1 and lies in W_2 .

Thus in all cases there is a general line lying in W_2 which is normal to W_1 .

The above theorem might also be stated in the form:

If a general threefold W_1 be normal to a general threefold W_2 , then W_2 is normal to W_1 .

SOME ANALOGIES.

Before proceeding with the next part of our subject we shall point out a few analogies which exist between an acceleration plane, a rotation threefold and the whole set of elements.

We have seen that: if P be an acceleration plane and A be any element in it there are two and only two optical lines passing through A and lying in P .

We have an analogue to this in the case of a rotation threefold.

We shall show that if W be a rotation threefold and a be any separation line in it there are two and only two optical planes containing a and lying in W .

In order to prove this: let O be any element in a .

Then, by Theorem 159, there is at least one general plane lying in W and passing through O to which a is normal.

Further, there cannot be more than one such general plane, for otherwise a would require to be normal to the rotation threefold W and would therefore intersect W contrary to the hypothesis that a lies in W .

Let P be this one general plane.

Then P cannot be a separation plane, for, since a is a separation line, this would require W to be a separation threefold, contrary to hypothesis.

Again, P cannot be an optical plane for this would require W to be an optical threefold, contrary to hypothesis.

Thus P must be an acceleration plane and so there are two and only two optical lines, say c_1 and c_2 , which pass through O and lie in P .

Thus a must be normal to both c_1 and c_2 and it cannot be normal to any other optical line passing through O and lying in W ; for such an optical line could not lie in P , and if a were normal to such an optical line in addition to c_1 and c_2 , it would require to be normal to W , which we know to be impossible.

But now c_1 and a lie in one optical plane, say R_1 , while c_2 and a lie in another optical plane, say R_2 .

Now R_1 and R_2 are the only optical planes in W which contain a ; for the existence of a third would require the existence of a third optical line passing through O , lying in W and normal to a , which, as we have seen, is impossible.

This proves the required result.

Again we have a corresponding result for the whole set of elements.

We shall show that if S be any separation plane there are two and only two optical threefolds containing S .

For let O be any element in S and let P be the one single acceleration plane which passes through O and is completely normal to S .

Further let c_1 and c_2 be the two generators of P which pass through O .

Then c_1 and c_2 are each normal to S , and accordingly c_1 and S determine one optical threefold, say W_1 , while c_2 and S determine another optical threefold, say W_2 .

Now W_1 and W_2 are the only optical threefolds which contain S , for the existence of a third would require the existence of a third optical line passing through O and normal to S .

But if there were three optical lines passing through O and normal to S , there would be more than one acceleration plane passing through O and completely normal to S , which we have seen is impossible.

Thus there are two and only two optical threefolds containing S , and so we see that we have here a certain analogy between an acceleration plane, a rotation threefold, and the whole set of elements.

It was pointed out in another part of this work that if W be a rotation threefold and A be any element in it, then there are an infinite number of optical lines which pass through A and lie in W .

It is easy to show that if a be any separation line, there are an infinite number of optical planes which contain a , although, as we have seen, there are only two in any one rotation threefold containing a .

Thus let O be any element in a and let W be the one single rotation threefold which passes through O and is normal to a .

Then there are an infinite number of optical lines passing through O and lying in W , and each of these must be normal to a .

Thus each of these optical lines and a determines an optical plane and all these latter must be distinct.

It follows that there are an infinite number of optical planes containing any separation line.

It is easy to show that if W be a rotation threefold and a be any optical line in it, then there is one and only one optical plane containing a and lying in W .

For let O be any element in a ; then, since there are an infinite number of optical lines passing through O and lying in W , there are an infinite number of acceleration planes lying in W and containing a .

Let P be any such acceleration plane.

Then by Theorem 160 there is one and only one general line, say b , passing through O and lying in W which is normal to P .

Then b must be normal to a , and so a and b determine an optical plane, say R , which lies in W .

Now R is the only optical plane which contains a and lies in W ; for let R' be any other optical plane containing a .

Then any element X lying in R' but not in a would be neither *before* nor *after* any element of R , and so X and R would lie in an optical threefold and could not lie in W .

This proves that R is the only optical plane containing a and lying in W .

If now we consider the whole set of elements we can easily show that if a be an optical line there is one and only one optical threefold containing a .

In order to prove this we have only to remember that a is normal to any optical threefold containing it and, by Theorem 162, if O be any element in a , there is one and only one optical threefold passing through O and normal to a .

Again, if P be an optical plane, there is one and only one optical threefold containing P ; for if A be any element which is neither *before* nor *after* any element of P , then P and A determine an optical threefold, say W .

Also W is the only optical threefold containing P , for otherwise we should have more than one optical threefold containing any optical line in P .

THEOREM 172.

If A, B, C, D be the corners of an optical parallelogram (AC being the inertia diagonal line) and if A, B', C, D' be the corners of a second optical parallelogram, while A', B', C', D' are the corners of a third optical parallelogram whose diagonal line $A'C'$ is conjugate to BD , then A', B, C', D will be the corners of a fourth optical parallelogram.

In order to prove this important theorem, we shall first prove the following lemma.

If O, C and C' be three distinct elements in an acceleration plane P such that OC and OC' are inertia lines while CC' is a separation line, and if further CC'' be another separation line intersecting OC' in C'' , and if M be the mean of C and C' while N is the mean of C and C'' , then if MO be conjugate to CC' we cannot have NO conjugate to CC'' .

It will be sufficient to consider the case where O is *before* C , since the case where O is *after* C is quite analogous.

Since CC' is a separation line, while OC' is an inertia line, and since O is *before* C it follows that O must also be *before* C' .

Let E, C, F, C' be the corners of an optical parallelogram in the acceleration plane P and let F' be *after* E .

Then FE is conjugate to CC' and intersects it in M and must therefore by hypothesis be identical with MO .

Now E must be *after* O , for in the first place E cannot be identical with O since EC' is an optical line while OC' is an inertia line.

Again, O cannot be *after* E for then we should have O *after* one element of the optical line EC' and *before* another element of it contrary to Theorem 12.

Thus since OE is an inertia line we must have E *after* O .

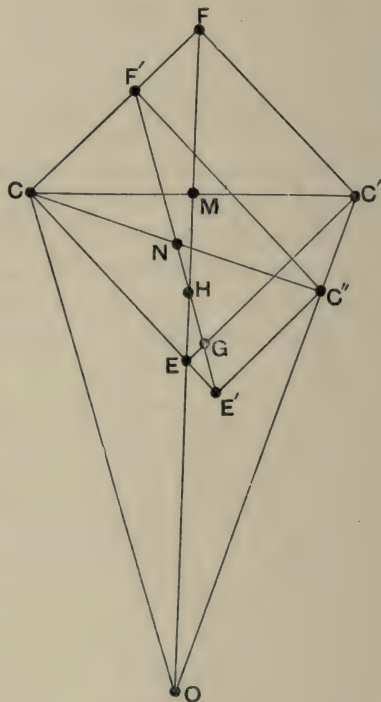


Fig. 38.

Now the element C'' is distinct from C' , and since C and C' lie in an inertia line we must have one *after* the other.

Suppose first that C' is *after* C'' .

Let the optical line through C'' parallel to $C'E$ intersect CE in E' and let the optical line through C'' parallel to $E'C$ intersect CF in F' .

Then E', C, F', C'' are the corners of an optical parallelogram, and, since CC'' is a separation line, $E'F'$ must be an inertia line conjugate to it and intersecting it in the element N .

But now $C''E'$ is a before-parallel of $C'E$ while $C''F'$ is a before-parallel of $C'F$.

Thus we must have E' *before* E and F' *before* F .

Now let the inertia line $E'F'$ intersect the optical line EC' in the element G .

Then E' is *before* E and is therefore in the β sub-set of E and so G must be in the α sub-set of E .

Thus G must be *after* E .

But since we have also F *after* F' it follows that EF and GF' intersect in an element, say H , which is between EC' and CF .

Thus H is linearly between E and F and is therefore *after* E .

But E is *after* O and therefore H is *after* O .

Thus the conjugate to CC'' through N in the acceleration plane P intersects MO in an element which is *after* O and so NO cannot be conjugate to CC'' .

This proves the lemma provided that C' is *after* C'' .

Next consider the case where C'' is *after* C' .

Suppose, if possible, that NO is conjugate to CC'' .

Then by the case already proved MO could not be conjugate to CC' , contrary to hypothesis, and so the lemma is proved in general.

We shall now make use of this lemma in order to prove the theorem.

We shall suppose that C is *after* A and C' *after* A' .

Now since the first and second optical parallelograms have the pair of opposite corners A and C in common it follows, by Theorem 59, that they have a common centre, say O .

Further, since the second and third optical parallelograms have the pair of opposite corners B' and D' in common, they have also the same centre O .

Thus AC and $A'C'$ intersect in the element O , and since they are both inertia lines they must lie in one acceleration plane, say P .

But C and C' are distinct elements lying in the α sub-sets of B' and

D' , and therefore C' is neither *before* nor *after* C , and, in an analogous way, A' is neither *before* nor *after* A .

Thus CC' and AA' are both separation lines.

Let M be the mean of C and C' .

Then B' , C and C' are three corners of an optical parallelogram having M as centre, while D' , C and C' are three corners of another optical parallelogram of which M is the centre.

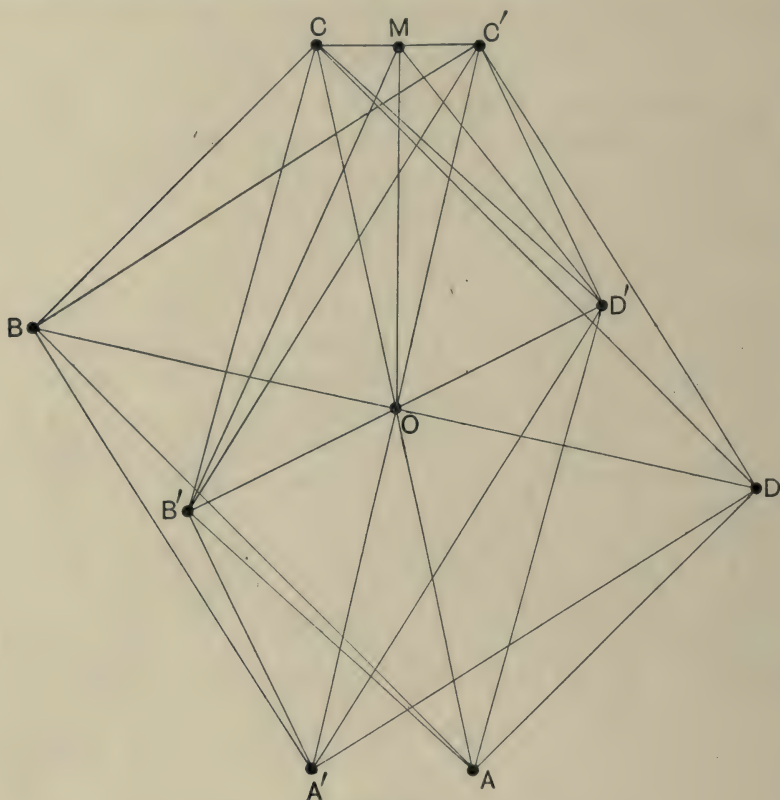


Fig. 39.

Further MB' and MD' must both be inertia lines and are each conjugate to CC' .

Thus, by Theorem 102, CC' is conjugate to every inertia line which passes through M and lies in the acceleration plane containing MB' and MD' .

But O is linearly between B' and D' while M is *after* both B' and D' but is not in the general line $B'D'$ and so, by Theorem 129 (b), MO is an inertia line.

Thus, since MO is in the acceleration plane containing MB' and MD' , it follows that CC' is conjugate to MO .

If now we consider the optical parallelogram having B and D as opposite corners and lying in the acceleration plane containing BD and $A'C'$ it follows, since O is the mean of B and D , that O must be the centre of this optical parallelogram.

Further, since by hypothesis $A'C'$ is conjugate to BD , it follows that the remaining two corners of this optical parallelogram must lie in $A'C'$.

Let A'' and C'' be these remaining corners and let C'' be *after* A'' .

Then just as CC' was shown to be a separation line, we may show that CC'' is a separation line, and if N be the mean of C and C'' we may show that CC'' is conjugate to NO , which may be proved to be an inertia line as was MO .

But if CC'' were distinct from CC' , our lemma shows that this would not be possible, and so CC'' must be identical with CC' .

Thus, since C'' lies in $A'C'$, it follows that C'' is identical with C' .

Similarly A'' is identical with A' and therefore A', B, C', D are the corners of an optical parallelogram as was to be proved.

THEORY OF CONGRUENCE.

We are now in a position to consider the problems of *congruence* and *measurement* in our system of geometry.

The first point to be examined is the *congruence* of pairs of elements, and we shall find that there are several cases which have to be considered separately.

Two distinct elements A and B will be spoken of briefly as a *pair* and will be denoted by the symbols (A, B) or (B, A) .

The order in which the letters are written will be taken advantage of in order to symbolize a certain correspondence between the elements of pairs, as we shall shortly explain.

Since any two distinct elements determine a general line, there will always be one general line associated with any given pair, but different pairs will be associated with the same general line.

If we set up a correspondence between the elements of a pair (A, B) and a pair (C, D) we might either take C to correspond to A and D to B , or else take D to correspond to A and C to B .

The first of these might be symbolized briefly by:

(A, B) corresponds to (C, D) ,

or

(B, A) corresponds to (D, C) .

The second might be symbolized by :

(A, B) corresponds to (D, C),
or (B, A) corresponds to (C, D).

If we consider the case of pairs which have a common element, say (A, B) and (A, C), and if

(A, B) corresponds to (A, C),

then the element A corresponds to itself and will be said to be *latent*.

Now the *congruence* of pairs is a correspondence which can be set up in a certain way between certain pairs lying in general lines of the same type.

In dealing with this subject it will be found convenient to have a systematic notation for optical parallelograms, so that we may be able to distinguish how the different corners are related.

If A, B, C, D be the corners of an optical parallelogram we shall use the notation $A\overline{BC}D$ when we wish to signify that the corners A and D lie in the inertia diagonal line and that A is *before* D, while B and C lie in the separation diagonal line so that the one is neither *before* nor *after* the other.

If O be the centre of the optical parallelogram $A\overline{BC}D$, then it is obvious that O will be *after* A and *before* D.

Definition. A pair (A, B) will be spoken of as an *optical pair*, an *inertia pair*, or a *separation pair* according as AB is an optical, an inertia, or a separation line.

We shall first give a definition of the congruence of inertia pairs having a latent element.

Definition. If $A_1\overline{BC}D_1$ and $A_2\overline{BC}D_2$ be optical parallelograms having the common pair of opposite corners B and C and the common centre O, then the inertia pair (O, D₁) will be said to be *congruent* to the inertia pair (O, D₂).

This will be written :

$$(O, D_1) (\equiv) (O, D_2).$$

Similarly the inertia pair (O, A₁) will be said to be *congruent* to the inertia pair (O, A₂).

If (O, D₁) be any inertia pair and a be any inertia line intersecting OD₁ in O, then the above definition enables us to show that there is one and only one element, say X, in a which is distinct from O and such that :

$$(O, D_1) (\equiv) (O, X).$$

For, by Theorem 105, there is at least one separation line, say c , which passes through O and is conjugate to both OD_1 and a .

Thus OD_1 and c determine an acceleration plane, say P_1 , while a and c determine an acceleration plane, say P_2 .

Now if D_1 be *after* O there is one single optical parallelogram in P_1 having O as centre and D_1 as one of its corners.

If A_1 be the corner opposite D_1 and if B and C be the remaining corners, this optical parallelogram will be $A_1\overline{BC}D_1$, where B and C will lie in c .

Again in the acceleration plane P_2 there will be one single optical parallelogram having B and C as a pair of opposite corners and O as centre.

If A_2 and D_2 be the remaining corners they will lie in a , and if D_2 be *after* A_2 , this optical parallelogram will be $A_2\overline{BC}D_2$.

Thus we may identify D_2 with X and can say that there is *at least one* element X lying in a and distinct from O and such that:

$$(O, D_1)(\equiv)(O, X).$$

We have now to show that the element X is unique in this respect in the general line a .

Let c' be any other separation line distinct from c which passes through O and is conjugate to both OD_1 and a .

Then OD_1 and c' determine an acceleration plane, say P_1' , while a and c' determine an acceleration plane, say P_2' .

There is one single optical parallelogram in P_1' , having A_1 and D_1 as a pair of opposite corners, and this optical parallelogram has also O as its centre.

If B' and C' be the remaining corners this optical parallelogram will be $A_1\overline{B'C'}D_1$.

But now we have the optical parallelograms $A_1\overline{B'C'}D_1$, $A_1\overline{BC}D_1$, $A_2\overline{BC}D_2$ and the diagonal line A_2D_2 of the last of these is conjugate to $B'C'$ and so it follows, by Theorem 172, that the elements A_2 , B' , D_2 , C' form the corners of a fourth optical parallelogram $A_2\overline{B'C'}D_2$.

Now $A_2\overline{B'C'}D_2$ will lie in the acceleration plane P_2' and will have O as its centre, and further $A_2\overline{B'C'}D_2$ is the only optical parallelogram which lies in P_2' and has B' and C' as a pair of opposite corners.

Thus the element D_2 or X is independent of the particular separation line passing through O and conjugate to both OD_1 and a , which we may select as the separation diagonal line of our optical parallelograms.

It follows that there is one and only one element X in a which is such that:

$$(O, D_1)(\equiv)(O, X).$$

The same result follows if D_1 be *before* O instead of *after* it.

Again if (O, D_1) , (O, D_2) and (O, D_3) be inertia pairs such that:

$$(O, D_1) (\equiv) (O, D_2)$$

and

$$(O, D_2) (\equiv) (O, D_3),$$

we may easily show that:

$$(O, D_1) (\equiv) (O, D_3).$$

In order to see this we have only to remember that whether the inertia lines OD_1 , OD_2 , OD_3 all lie in one acceleration plane or in one rotation threefold, there must be at least one general line passing through O and normal to all three.

Since only a separation line can be normal to an inertia line, this separation line will be conjugate to OD_1 , OD_2 and OD_3 , and if we call it c , then OD_1 and c will determine an acceleration plane, say P_1 , OD_2 and c will determine an acceleration plane, say P_2 , and OD_3 and c will determine an acceleration plane, say P_3 .

Now in P_1 there will be one single optical parallelogram having O as centre and D_1 as one of its corners, while in P_2 there will be one single optical parallelogram having O as centre and D_2 as one of its corners, and finally in P_3 there will be one single optical parallelogram having O as centre and D_3 as one of its corners.

Since $(O, D_1) (\equiv) (O, D_2)$ and $(O, D_2) (\equiv) (O, D_3)$ these three optical parallelograms will have a common pair of opposite corners, and so it follows from the definition that:

$$(O, D_1) (\equiv) (O, D_3).$$

Thus *for inertia pairs having a latent element, the relation of congruence is a transitive relation.*

It is to be observed that if (O, D_1) be an inertia pair we may write:

$$(O, D_1) (\equiv) (O, D_1),$$

or an inertia pair is to be regarded as congruent to itself.

We shall next consider the congruence of separation pairs having a latent element.

This case differs somewhat from the one we have considered.

While two intersecting inertia lines always lie in an acceleration plane, two intersecting separation lines may lie either in a separation plane, an optical plane, or an acceleration plane.

An inertia line can only be conjugate to two intersecting separation lines if these lie in a separation plane, as follows from Theorem 99.

Thus if we were to give a definition of the congruence of separation pairs having a latent element which was strictly analogous to that given for inertia pairs, such a definition would be incomplete.

It is however possible, by a slight modification, to give a definition which will hold for all cases.

In order to avoid complication we shall first explain what we mean by an inertia pair being "conjugate" to a separation pair or a separation pair being "conjugate" to an inertia pair.

Definition. If $AB\overline{CD}$ be an optical parallelogram and O be its centre, then the inertia pairs (O, D) and (O, A) will be spoken of as *conjugates* to the separation pairs (O, B) and (O, C) and also conversely.

The pair (O, D) will be called an *after-conjugate* to the pairs (O, B) , (O, C) , while (O, A) will be called a *before-conjugate* to the pairs (O, B) , (O, C) .

Further, either of the separation pairs (O, B) , (O, C) will be called an *after-conjugate* to (O, A) and a *before-conjugate* to (O, D) .

Now we know that there are an infinite number of acceleration planes which contain any given separation line, and so there are always inertia pairs which are conjugate to any given separation pair.

Knowing this we can give the following definition of the "congruence" of separation pairs having a latent element.

Definition. If (O, B_1) and (O, B_2) be separation pairs and if (O, D_1) and (O, D_2) be inertia pairs which are after-conjugates to (O, B_1) and (O, B_2) respectively, then if $(O, D_1) (\equiv) (O, D_2)$ we shall say that (O, B_1) is *congruent* to (O, B_2) and shall write this:

$$(O, B_1) \{ \equiv \} (O, B_2).$$

If (O, D_1') be any inertia pair which is an after-conjugate to (O, B_1) , but is distinct from (O, D_1) , then it is obvious by definition that:

$$(O, D_1) (\equiv) (O, D_1').$$

But since

$$(O, D_1) (\equiv) (O, D_2),$$

and, since these are inertia pairs, it follows that:

$$(O, D_1') (\equiv) (O, D_2).$$

Thus the congruence of (O, B_1) to (O, B_2) is independent of the particular after-conjugate to (O, B_1) which we may select, and similarly, it is independent of the particular after-conjugate to (O, B_2) which we may select.

Again if (O, B_1) , (O, B_2) and (O, B_3) be separation pairs such that:

$$(O, B_1) \{\equiv\} (O, B_2),$$

and

$$(O, B_2) \{\equiv\} (O, B_3),$$

we may easily show that:

$$(O, B_1) \{\equiv\} (O, B_3).$$

In order to prove this, let (O, D_1) , (O, D_2) and (O, D_3) be inertia pairs which are after-conjugates to (O, B_1) , (O, B_2) and (O, B_3) respectively.

Then we must have

$$(O, D_1) (\equiv) (O, D_2),$$

and

$$(O, D_2) (\equiv) (O, D_3),$$

and, since these are inertia pairs, it follows, as previously shown, that:

$$(O, D_1) (\equiv) (O, D_3).$$

Thus, by the definition:

$$(O, B_1) \{\equiv\} (O, B_3),$$

and so, *for separation pairs having a latent element, the relation of congruence is a transitive relation.*

Again, if (O, B) be any separation pair and a be any separation line passing through O , there are two and only two elements, say X_1 and Y_1 , in a which are distinct from O and such that:

$$(O, B) \{\equiv\} (O, X_1),$$

and

$$(O, B) \{\equiv\} (O, Y_1).$$

This may be easily shown as follows.

Let (O, D) be any inertia pair which is an after-conjugate to (O, B) and let b be any inertia line which passes through O and is conjugate to a .

Then, as we have already seen, there is one and only one element, say D_1 , lying in b and distinct from O and such that:

$$(O, D) (\equiv) (O, D_1).$$

But now a and b determine an acceleration plane and in this acceleration plane there is one and only one optical parallelogram having O as centre and D_1 as one of its corners.

If this optical parallelogram be $A_1\bar{B}_1\bar{C}_1D_1$, then the elements B_1 and C_1 will lie in a and the inertia pair (O, D_1) will be an after-conjugate to each of the separation pairs (O, B_1) and (O, C_1) .

Thus since $(O, D) (\equiv) (O, D_1)$ it follows that:

$$(O, B) \{\equiv\} (O, B_1)$$

and

$$(O, B) \{\equiv\} (O, C_1).$$

Again if there were any other element, say B_2 , lying in a and distinct from both B_1 and C_1 and such that we had

$$(O, B) \{ \equiv \} (O, B_2),$$

then there would be an element, say D_2 , lying in b and such that (O, D_2) was an after-conjugate to (O, B_2) .

Since B_2 is supposed distinct from both B_1 and C_1 , therefore D_2 would require to be distinct from D_1 .

But since we have supposed $(O, B) \{ \equiv \} (O, B_2)$, therefore we should have $(O, D) (\equiv) (O, D_2)$ and so we should have the two distinct elements D_1 and D_2 lying in the inertia line b and such that:

$$(O, D) (\equiv) (O, D_1) \text{ and } (O, D) (\equiv) (O, D_2),$$

which we have already shown to be impossible.

Thus we may identify B_1 with X_1 and C_1 with Y_1 and say that there are two and only two elements X_1 and Y_1 lying in a and distinct from O and such that:

$$(O, B) \{ \equiv \} (O, X_1) \text{ and } (O, B) \{ \equiv \} (O, Y_1).$$

If \overline{ABCD} be an optical parallelogram and O be its centre, we observe that according to our definitions we have

$$(O, B) \{ \equiv \} (O, C),$$

but not

$$(O, A) (\equiv) (O, D).$$

The reason why we make this distinction is that in the separation pairs we have O neither *before* nor *after* B and also O neither *before* nor *after* C , while in the inertia pairs we have O *after* A and O *before* D .

Thus in the first case the relations are alike in respect of *before* and *after*, while in the second case the relations are different.

The question now arises as to the "congruence" of optical pairs.

In this case constructions such as those by which we defined the congruence of inertia and separation pairs having a latent element, entirely fail and there is nothing at all analogous to them.

We are thus led to regard optical pairs as not determinately comparable with one another in respect of congruence, except when they lie in the same, or in parallel optical lines.

As regards the "congruence" of pairs lying in the same general line, we have as yet given no definition, except for the very special case of inertia or separation pairs having a latent element; while no definition whatever has been given of the "congruence" of pairs lying in parallel general lines.

A definition covering all these omitted cases can be given, which applies to all three types of pair.

We must first however define what we mean when we say that one pair is opposite to another.

Definition. A pair (A, B) will be said to be *opposite* to a pair (C, D) if and only if the elements A, B, C, D form the corners of a general parallelogram in such a way that AB and CD are one pair of opposite sides, while AC and BD are the other pair of opposite sides.

This will be denoted by the symbols

$$(A, B) \square (C, D).$$

It will be observed that the use of the symbol \square implies that the pairs (A, B) and (C, D) lie in distinct general lines which are parallel to one another.

If however we have

$$(A, B) \square (C, D),$$

and

$$(E, F) \square (C, D),$$

then the pairs (A, B) and (E, F) may lie either in the same or in parallel general lines.

If (A, B) and (E, F) do not lie in the same general line, it follows from Theorem 127 that we may write

$$(A, B) \square (E, F).$$

We have now to prove the following theorem:

THEOREM 173.

If (A, B) , (A', B') and (C, D) be pairs such that:

$$(A, B) \square (C, D),$$

and

$$(A', B') \square (C, D),$$

and if (C', D') be any other pair such that:

$$(A, B) \square (C', D'),$$

and which does not lie in the general line $A'B'$, then we shall also have

$$(A', B') \square (C', D').$$

We shall first consider the case where (A, B) and (A', B') do not lie in one general line.

In this case since

$$(A, B) \square (C, D),$$

and

$$(A', B') \square (C, D),$$

it follows by Theorem 127 that:

$$(A', B') \square (A, B).$$

But

$$(C', D') \square (A, B)$$

by hypothesis, and so, since (C', D') and (A', B') do not lie in one general line, it follows that:

$$(A', B') \square (C', D').$$

Next consider the case where (A, B) and (A', B') lie in one general line.

There are two sub-cases of this:

(1) (C, D) and (C', D') do not lie in one general line.

(2) (C, D) and (C', D') do lie in one general line.

Consider first sub-case (1).

Here since $(C, D) \square (A, B)$,

and $(C', D') \square (A, B)$,

and since (C, D) and (C', D') do not lie in one general line, it follows that:

$$(C', D') \square (C, D).$$

But $(A', B') \square (C, D)$,

and so, since (C', D') and (A', B') do not lie in one general line, it follows that:

$$(A', B') \square (C', D').$$

Next consider sub-case (2).

Let E be any element in the general line AC' distinct from both A and C' and let a general line through E parallel to AB intersect $D'B$ in the element F .

Then we shall have*

$$(E, F) \square (A, B),$$

and also $(E, F) \square (C', D')$.

But now since E is distinct from A and also from C' it follows that the general line EF must be distinct from the general line containing (A, B) and (A', B') and must also be distinct from the general line containing (C', D') and (C, D) .

Thus since $(E, F) \square (A, B)$,

and $(C, D) \square (A, B)$,

and since (E, F) and (C, D) do not lie in one general line, it follows that:

$$(E, F) \square (C, D).$$

Also since $(A', B') \square (C, D)$,

and since (E, F) and (A', B') do not lie in one general line, it follows that:

$$(A', B') \square (E, F).$$

But $(C', D') \sqsubset (E, F)$,

and since (A', B') and (C', D') lie respectively in the distinct general lines AB and CD , it follows that:

$$(A', B') \sqsubset (C', D').$$

Thus the theorem holds in all cases.

We are now in a position to introduce the following definition:

Definition. A pair (A, B) will be said to be *co-directionally congruent* to a pair (A', B') provided a pair (C, D) exists such that:

$$(A, B) \sqsubset (C, D),$$

and

$$(A', B') \sqsubset (C, D).$$

The theorem just proved shows that we are at liberty to replace the pair (C, D) by any other pair (C', D') such that:

$$(A, B) \sqsubset (C', D'),$$

provided (C', D') does not lie in the general line $A'B'$.

It is evident that $(A, B) \sqsubset (C, D)$ implies that (A, B) is co-directionally congruent to (C, D) , but (A, B) being co-directionally congruent to (C, D) does not imply that $(A, B) \sqsubset (C, D)$, since (A, B) and (C, D) might lie in the same general line.

It is also obvious that: (A, B) is co-directionally congruent to (A, B) .

We shall ultimately represent co-directional congruence by the same symbol \equiv as we shall use for the other cases of congruence, but when we wish to make it clear that the congruence is co-directional we shall use the symbol $|\equiv|$.

Thus we see that: $(A, B) \sqsubset (C, D)$ implies $(A, B) |\equiv| (C, D)$, but $(A, B) |\equiv| (C, D)$ does not imply $(A, B) \sqsubset (C, D)$, except when AB and CD are distinct general lines.

We have next to show that if

$$(A, B) |\equiv| (C, D),$$

and

$$(C, D) |\equiv| (E, F),$$

then must

$$(A, B) |\equiv| (E, F).$$

This is easily proved; for if a be any general line parallel to AB but distinct from CD and EF and therefore also parallel to them, we may select any pair (G, H) in a , such that:

$$(A, B) \sqsubset (G, H) \dots \dots \dots (1).$$

Then since

$$(A, B) |\equiv| (C, D),$$

it follows, by Theorem 173, that:

$$(C, D) \sqsubset (G, H).$$

Similarly since $(C, D) \equiv (E, F)$,

it follows that: $(E, F) \sqsubset (G, H) \dots\dots\dots(2)$.

Thus from (1) and (2) it follows that:

$$(A, B) \equiv (E, F),$$

and so we see that: *the relation of co-directional congruence of pairs is a transitive relation.*

If $(A, B) \sqsubset (C, D)$ and if B be *after* A then it is easy to see that D must be *after* C .

In the first place AB must be either an optical or inertia line and, since CD is parallel to AB , it follows that CD must be the same type of general line as AB .

Suppose first that AB is an optical line.

Then C could not be *after* D , for then by Theorem 57 or Theorem 91 AC and BD would intersect, contrary to the hypothesis that they are parallel.

Thus, since C and D are distinct, and since CD is an optical line, it follows that D must be *after* C .

Next suppose that AB is an inertia line.

Then AB and CD must lie in an acceleration plane, say P .

If AC and BD should happen to be optical lines then, since B is *after* A , it follows that BD would be an after-parallel of AC and so, since CD is an inertia line, it would follow that D must be *after* C .

Next suppose that AC and BD are not optical lines.

Let AE and BE be generators of P of opposite sets passing through A and B respectively and intersecting in E .

Let CF be an optical line through C parallel to AE and let it intersect the general line through E parallel to AC in F .

Then EF must be parallel to BD and so, by Theorem 127, DF must be parallel to BE and therefore must be an optical line.

But now since B is *after* A we must have E *after* A and B *after* E .

Thus by the first case we must have F *after* C and D *after* F and therefore D *after* C as was to be proved.

Thus in all cases if B be *after* A we must have D *after* C and similarly if B be *before* A we must have D *before* C .

It follows directly from this that if

$$(A, B) \sqsubset (C, D),$$

and

$$(A', B') \sqsubset (C, D),$$

then if B be *after* A we must have D *after* C and therefore B' *after* A' .

Thus if $(A, B) \equiv (A', B')$ and if B be after A we must have B' after A' while if B be before A we must have B' before A' .

Again if three corners of a general parallelogram A', C and D be given and if we know that two of the side lines are $A'C$ and CD , then the general parallelogram is uniquely determined.

If then any pair (A', X) be co-directionally congruent to a pair (A, B) , where A, B and A' are given, it is easy to see that X is uniquely determinate, provided we know that A' corresponds to A .

For let a be a general line parallel to AB , but which does not pass through A' , and let (C, D) be any pair in a such that:

$$(A, B) \square (C, D).$$

Then there is one single general parallelogram having A', C and D as three of its corners and $A'C$ and CD as two of its side lines.

If B' be the remaining corner we shall have

$$(A', B') \square (C, D),$$

and so

$$(A', B') \equiv (A, B).$$

Thus X must be identified with B' , which is a definite element.

THEOREM 174.

If (O_1, A_1) and (O_2, A_2) be inertia pairs such that:

$$(O_1, A_1) \equiv (O_2, A_2),$$

and if (O_1, B_1) be any separation pair which is conjugate to (O_1, A_1) , then there is a separation pair, say (O_2, B_2) , which is conjugate to (O_2, A_2) and such that:

$$(O_1, B_1) \equiv (O_2, B_2).$$

It is evident that (O_1, A_1) and (O_1, B_1) must lie in an acceleration plane, say P_1 .

Since the inertia line O_2A_2 must be either parallel to the inertia line O_1A_1 , or else identical with it, it follows that O_2A_2 must either lie in P_1 or in an acceleration plane parallel to P_1 .

We shall first consider the case where O_2A_2 lies in an acceleration plane P_2 parallel to P_1 .

If now we take the one single optical parallelogram in P_1 having O_1 as centre and A_1 as one of its corners, then B_1 will be another corner.

If (O_1, B_1) be an after-conjugate to (O_1, A_1) we may take this optical parallelogram to be $A_1\overline{B_1C_1}D_1$, while if (O_1, B_1) be a before-conjugate to (O_1, A_1) we may take the optical parallelogram to be $D_1\overline{B_1C_1}A_1$.

Now the acceleration plane P_1 and the general line O_1O_2 determine a general threefold containing P_2 and so, as we have already seen, if

through any element of P_1 distinct from O_1 a general line be taken parallel to O_1O_2 then this general line will intersect P_2 .

Now through the elements A_1, B_1, C_1 and D_1 let general lines be taken parallel to O_1O_2 and let these intersect P_2 in the elements A_2, B_2, C_2 and D_2 respectively.

Then any two of the general lines $O_1O_2, A_1A_2, B_1B_2, C_1C_2, D_1D_2$ are parallel to one another and therefore any two of them lie in a general plane.

Since however the elements A_1, O_1 and D_1 lie in one general line, the three general lines A_1A_2, O_1O_2 and D_1D_2 lie in one general plane, and since the elements B_1, O_1 and C_1 lie in one general line, the three general lines B_1B_2, O_1O_2 and C_1C_2 lie in one general plane.

Thus the elements A_2, O_2 and D_2 lie in one general line parallel to the general line containing A_1, O_1 and D_1 , while the elements B_2, O_2

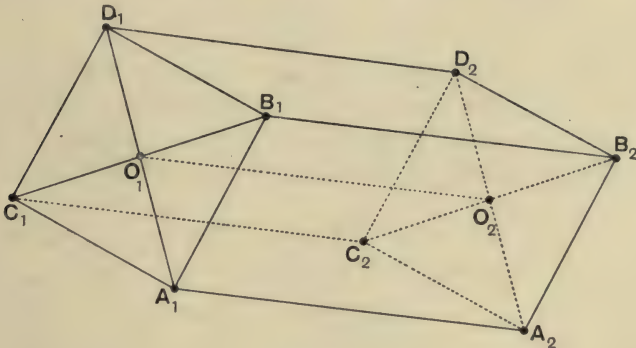


Fig. 40.

and C_2 lie in another general line parallel to that containing B_1, O_1 and C_1 .

Further the general lines $A_2B_2, A_2C_2, B_2D_2, C_2D_2$ must be respectively parallel to $A_1B_1, A_1C_1, B_1D_1, C_1D_1$ and, since these latter are all optical lines, it follows that $A_2B_2, A_2C_2, B_2D_2, C_2D_2$ are all optical lines.

Thus A_2, B_2^*, C_2, D_2 form the corners of an optical parallelogram having O_2 as centre.

Further the diagonal line A_2D_2 is an inertia line, while the diagonal line B_2C_2 must be a separation line.

Thus the separation pair (O_2, B_2) is conjugate to the inertia pair (O_2, A_2) , and, since O_2 is *after* or *before* A_2 according as O_1 is *after* or *before* A_1 , it follows that (O_2, B_2) is an after- or before-conjugate to (O_2, A_2) according as (O_1, B_1) is an after- or before-conjugate to (O_1, A_1) .

Also we have $(O_1, B_1) \square (O_2, B_2)$
 and so $(O_1, B_1) \equiv (O_2, B_2)$.

This proves the theorem provided O_2A_2 does not lie in P_1 .

Consider next the case where O_2A_2 does lie in P_1 .

Let P' be any acceleration plane parallel to P_1 and let (O', A') be any inertia pair in P' such that:

$$(O_1, A_1) \square (O', A').$$

Then, by Theorem 173, since $(O_1, A_1) \equiv (O_2, A_2)$ we must have

$$(O_2, A_2) \square (O', A').$$

Thus, by the case already proved, it follows that there is an optical parallelogram lying in P' which has O' as centre and A' as one of its corners and such that, if we denote it by $A'B'C'D'$ or $D'B'C'A'$ (according as the optical parallelogram in P_1 is $A_1\overline{B_1C_1}D_1$ or $D_1\overline{B_1C_1}A_1$), then:

$$(O_1, B_1) \square (O', B').$$

But in a similar manner we can show that there is an optical parallelogram lying in P_1 which has O_2 as centre and A_2 as one of its corners and such that, if we denote it by $A_2\overline{B_2C_2}D_2$ or $D_2\overline{B_2C_2}A_2$ (according as the optical parallelogram in P' is $A'B'C'D'$ or $D'B'C'A'$), then:

$$(O_2, B_2) \square (O', B').$$

Thus it follows from definition that:

$$(O_1, B_1) \equiv (O_2, B_2).$$

Further (O_2, B_2) is conjugate to (O_2, A_2) and will be an after- or before-conjugate to (O_2, A_2) according as (O_1, B_1) is an after- or before-conjugate to (O_1, A_1) .

Thus the theorem holds in all cases.

THEOREM 175.

If (O_1, A_1) and (O_2, A_2) be separation pairs such that:

$$(O_1, A_1) \equiv (O_2, A_2)$$

and if (O_1, B_1) be any inertia pair which is conjugate to (O_1, A_1) then there is an inertia pair, say (O_2, B_2) , which is conjugate to (O_2, A_2) and such that:

$$(O_1, B_1) \equiv (O_2, B_2).$$

The proof of this theorem is quite analogous to that of Theorem 174.

Also it will be seen that (O_2, B_2) will be a before- or after-conjugate to (O_2, A_2) according as (O_1, B_1) is a before- or after-conjugate to (O_1, A_1) .

We have now to prove certain theorems involving both the co-directional congruence of pairs and the congruence of pairs having a latent element.

We shall make use of the symbols (\equiv) , $\{\equiv\}$ and $|\equiv|$ in the manner already explained in order to show clearly the types of congruence to which we refer.

THEOREM 176.

If (O_1, A_1) , (O_1, B_1) and (O_2, A_2) be inertia pairs such that:

$$(O_1, A_1) (\equiv) (O_1, B_1)$$

and

$$(O_1, A_1) |\equiv| (O_2, A_2),$$

then there is an inertia pair (O_2, B_2) such that:

$$(O_2, A_2) (\equiv) (O_2, B_2)$$

and

$$(O_1, B_1) |\equiv| (O_2, B_2).$$

Let c be any separation line passing through O_1 and normal to both the inertia lines O_1A_1 and O_1B_1 , and let C_1 be an element in c such that the separation pair (O_1, C_1) is conjugate to (O_1, A_1) .

Then since $(O_1, A_1) (\equiv) (O_1, B_1)$ it follows that (O_1, C_1) must also be conjugate to (O_1, B_1) .

But since $(O_1, A_1) |\equiv| (O_2, A_2)$ it follows, by Theorem 174, that there is a separation pair, say (O_2, C_2) , which is conjugate to (O_2, A_2) and such that:

$$(O_1, C_1) |\equiv| (O_2, C_2).$$

But now, by Theorem 175, since (O_1, B_1) is conjugate to (O_1, C_1) , it follows that there is an inertia pair, say (O_2, B_2) , which is conjugate to (O_2, C_2) and such that:

$$(O_1, B_1) |\equiv| (O_2, B_2).$$

But now since $(O_1, A_1) (\equiv) (O_1, B_1)$ and these are inertia pairs we must have A_1 and B_1 either both *after* O_1 or both *before* O_1 .

Further A_2 must be *after* or *before* O_2 according as A_1 is *after* or *before* O_1 , while B_2 must be *after* or *before* O_2 according as B_1 is *after* or *before* O_1 .

Thus A_2 and B_2 are either both *after* O_2 or both *before* O_2 .

Since therefore (O_2, C_2) is conjugate to both (O_2, A_2) and (O_2, B_2) it follows that:

$$(O_2, A_2) (\equiv) (O_2, B_2).$$

Thus the theorem is proved.

THEOREM 177.

If (O_1, A_1) , (O_1, B_1) and (O_2, A_2) be separation pairs such that:

$$(O_1, A_1) \{ \equiv \} (O_1, B_1)$$

and

$$(O_1, A_1) | \equiv | (O_2, A_2),$$

then there is a separation pair (O_2, B_2) such that:

$$(O_2, A_2) \{ \equiv \} (O_2, B_2)$$

and

$$(O_1, B_1) | \equiv | (O_2, B_2).$$

Let (O_1, D_1) and (O_1, E_1) be inertia pairs which are after-conjugates to (O_1, A_1) and (O_1, B_1) respectively.

Then since $(O_1, A_1) \{ \equiv \} (O_1, B_1),$

we must have $(O_1, D_1) (\equiv) (O_1, E_1).$

But now since $(O_1, A_1) | \equiv | (O_2, A_2),$

and since (O_1, D_1) is an inertia pair which is an after-conjugate to (O_1, A_1) , it follows, by Theorem 175, that there is an inertia pair, say (O_2, D_2) , which is an after-conjugate to (O_2, A_2) and such that:

$$(O_1, D_1) | \equiv | (O_2, D_2).$$

But now (O_1, D_1) , (O_1, E_1) and (O_2, D_2) are inertia pairs such that:

$$(O_1, D_1) (\equiv) (O_1, E_1)$$

and

$$(O_1, D_1) | \equiv | (O_2, D_2)$$

and so, by Theorem 176, there is a pair (O_2, E_2) such that:

$$(O_2, D_2) (\equiv) (O_2, E_2)$$

and

$$(O_1, E_1) | \equiv | (O_2, E_2).$$

Since however (O_1, B_1) is a separation pair which is a before-conjugate to the inertia pair (O_1, E_1) , it follows, by Theorem 174, that there is a separation pair, say (O_2, B_2) , which is a before-conjugate to (O_2, E_2) and such that:

$$(O_1, B_1) | \equiv | (O_2, B_2).$$

But since (O_2, D_2) and (O_2, E_2) are after-conjugates to (O_2, A_2) and (O_2, B_2) respectively, and since

$$(O_2, D_2) (\equiv) (O_2, E_2),$$

it follows by definition that:

$$(O_2, A_2) \{ \equiv \} (O_2, B_2).$$

Thus the theorem is proved.

THEOREM 178.

If (A, B) and (A, C) be inertia pairs such that:

$$(A, B) (\equiv) (A, C),$$

then there is an inertia pair (C, D) such that:

$$(C, A) (\equiv) (C, D)$$

and

$$(B, A) | \equiv | (C, D).$$

Let a be any separation line which passes through A and is normal to both AB and AC .

Let A_1 be an element in a such that the separation pair (A, A_1) is conjugate to the inertia pair (A, C) .

Then since $(A, B) (\equiv) (A, C)$,

it follows that (A, A_1) is also conjugate to (A, B) .

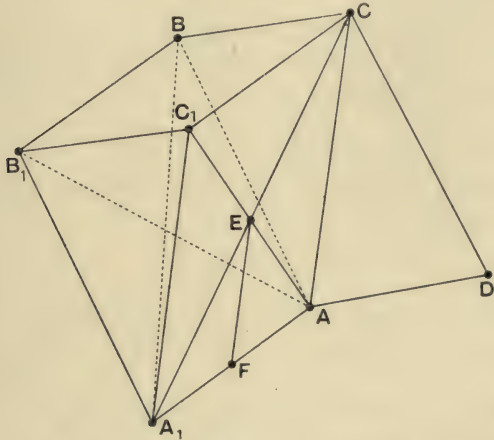


Fig. 41.

Let the general line through C parallel to AA_1 intersect the general line through A_1 parallel to AC in the element C_1 , and let the general line through B parallel to AA_1 intersect the general line through A_1 parallel to AB in the element B_1 .

Thus $(C, C_1) \square (A, A_1)$

and $(B, B_1) \square (A, A_1)$,

and therefore $(B, B_1) | \equiv | (C, C_1)$.

Now since (A, A_1) is conjugate to (A, C) it follows that A_1C is an optical line and it is easy to show that AC_1 is also an optical line as follows.

Since A_1C and AC_1 are diagonal lines of the general parallelogram whose corners are A, A_1, C_1, C , it follows that they intersect in an element, say E , which is the mean of A_1 and C .

If F be the mean of A_1 and A , then EF is parallel to CA and therefore EF is conjugate to the separation line AA_1 .

Thus A, A_1 and E are three corners of an optical parallelogram whose centre F lies in AA_1 and therefore AE (that is AC_1) is an optical line.

Similarly, since (A, A_1) is conjugate to (A, B) , it follows that AB_1 is an optical line.

But since CC_1 and BB_1 are parallel to AA_1 we have CC_1 conjugate to CA , and BB_1 conjugate to BA .

Thus the inertia pairs (C, A) and (B, A) are conjugate to the separation pairs (C, C_1) and (B, B_1) respectively.

But now since $(B, B_1) \equiv (C, C_1)$,

and since (B, A) is an inertia pair which is conjugate to (B, B_1) , it follows, by Theorem 175, that there is an inertia pair, say (C, D) , which is conjugate to (C, C_1) and such that:

$$(B, A) \equiv (C, D).$$

But from this it follows that D must be *before* or *after* C according as A is *before* or *after* B .

Since however $(A, B) \equiv (A, C)$,

we have A *before* or *after* B according as A is *before* or *after* C .

Thus we must have D *before* or *after* C according as A is *before* or *after* C .

Since therefore the inertia pairs (C, A) and (C, D) are both conjugate to the separation pair (C, C_1) , it follows that:

$$(C, A) \equiv (C, D).$$

Thus the theorem is proved.

THEOREM 179.

If (A, B) and (A, C) be separation pairs such that:

$$(A, B) \{ \equiv \} (A, C),$$

then there is a separation pair (C, D) such that:

$$(C, A) \{ \equiv \} (C, D)$$

and

$$(B, A) \equiv (C, D).$$

Let a be any inertia line which passes through A and is normal to AB , and let a' be any inertia line which passes through A and is normal to AC .

Let A_1 be an element in a such that the inertia pair (A, A_1) is an after-conjugate to the separation pair (A, B) and let A' be an element in a' such that the inertia pair (A, A') is an after-conjugate to the separation pair (A, C) .

Then since $(A, B) \{ \equiv \} (A, C)$

it follows that: $(A, A_1) (\equiv) (A, A')$.

Let the general line through C parallel to AA' intersect the general line through A' parallel to AC in the element C' and let the general

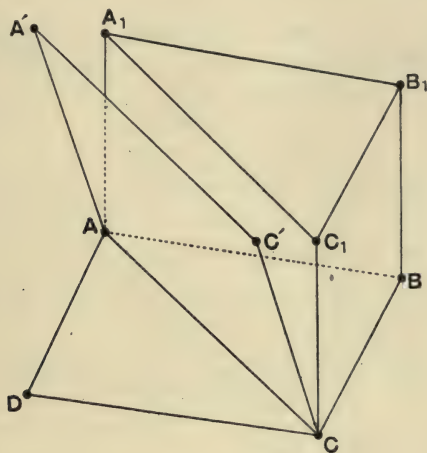


Fig. 42.

line through B parallel to AA_1 intersect the general line through A_1 parallel to AB in the element B_1 .

Then $(C, C') \sqsubseteq (A, A')$

and $(B, B_1) \sqsubseteq (A, A_1)$,

and so we may write $(C, C') \equiv (A, A')$

and $(B, B_1) \models (A, A_1)$.

But since (A, A') , (A, A_1) and (C, C') are inertia pairs such that:

$$(A, A') (\equiv) (A, A_1)$$

and $(A, A') \equiv (C, C')$,

therefore, by Theorem 176, there is an inertia pair (C, C_1) such that:

$$(C, C') (\equiv) (C, C_1)$$

and

$$(A, A_1) | \equiv | (C, C_1).$$

Thus since

$$(B, B_1) | \equiv | (A, A_1),$$

it follows that:

$$(B, B_1) | \equiv | (C, C_1).$$

But now we may show in the manner employed in the last theorem that since (A, A_1) is an after-conjugate to (A, B) , therefore (B, B_1) is an after-conjugate to (B, A) , and since (A, A') is an after-conjugate to (A, C) therefore (C, C') is an after-conjugate to (C, A) .

Since however we have the inertia pairs (B, B_1) and (C, C_1) such that:

$$(B, B_1) | \equiv | (C, C_1),$$

and since (B, A) is a separation pair which is conjugate to (B, B_1) , it follows, by Theorem 174, that there is a separation pair, say (C, D) , which is conjugate to (C, C_1) and such that:

$$(B, A) | \equiv | (C, D).$$

But now (A, A_1) is an after-conjugate to (A, B) and so A_1 is *after* A .

Thus since

$$(A, A_1) | \equiv | (C, C_1)$$

it follows that C_1 is *after* C , and so since (C, C_1) is conjugate to (C, D) it must be an after-conjugate.

But (C, C') is an after-conjugate to (C, A) and so since

$$(C, C') (\equiv) (C, C_1),$$

it follows from the definition that:

$$(C, A) \{ \equiv \} (C, D).$$

Thus the theorem is proved.

THEOREM 180.

(1) *If A, B and C be three distinct elements and the pairs (A, B) and (B, C) be such that:*

$$(A, B) | \equiv | (B, C),$$

then B is the mean of A and C .

(2) *If A, B and C be three distinct elements such that B is the mean of A and C , then the pairs (A, B) and (B, C) are such that:*

$$(A, B) | \equiv | (B, C).$$

First suppose that:

$$(A, B) | \equiv | (B, C).$$

Then by the definition of co-directional congruence there must be a pair, say (D, E) , such that:

$$(A, B) \square (D, E)$$

and

$$(B, C) \square (D, E).$$

Now since the pairs (A, B) and (B, C) have a common element B they cannot lie in parallel general lines and so must lie in the same general line.

Then BE and CD must be the diagonal lines of the general parallelogram whose corners are B, C, D and E and so BE and CD must intersect in an element F which is the mean of D and C .

But D does not lie in the general line AC , and so since BF is parallel to AD it follows, by Theorem 81, 92, or 116, that B is the mean of A and C .

Next, to prove the second part of the theorem suppose that B is the mean of A and C .

Let (D, E) be any pair such that:

$$(B, C) \square (D, E).$$

Then the diagonal lines BE and CD of the general parallelogram, whose corners are B, C, D and E , must intersect in an element F which is the mean of D and C .

But since D does not lie in the general line AC it follows, by Theorem 81, 92, or 116, that BF (that is BE) is parallel to AD .

Thus since also AB is parallel to DE it follows that:

$$(A, B) \square (D, E).$$

Thus by the definition of co-directional congruence we have

$$(A, B) || (B, C).$$

Thus both parts of the theorem are proved.

We are now in a position to introduce general definitions of the congruence of inertia and separation pairs.

This is done by combining co-directional congruence with congruence in which an element is latent, in the following manner.

Definition. An inertia pair (A_1, B_1) will be said to be *congruent* to an inertia pair (A_2, B_2) provided an inertia pair (A_2, C_2) exists such that:

$$(A_1, B_1) || (A_2, C_2)$$

and

$$(A_2, B_2) (\equiv) (A_2, C_2).$$

Definition. A separation pair (A_1, B_1) will be said to be *congruent* to a separation pair (A_2, B_2) provided a separation pair (A_2, C_2) exists such that:

$$(A_1, B_1) \mid \equiv \mid (A_2, C_2)$$

and

$$(A_2, B_2) \{ \equiv \} (A_2, C_2).$$

We shall denote the generalized congruence of inertia or of separation pairs by the symbol \equiv , thus:

$$(A_1, B_1) \equiv (A_2, B_2).$$

We shall also use the same symbol to denote the congruence of optical pairs, but in the latter case it is to be regarded as simply equivalent to the symbol $\mid \equiv \mid$, since the only congruence of optical pairs is taken to be co-directional.

Let us consider now two inertia pairs (A_1, B_1) and (A_2, B_2) such that:

$$(A_1, B_1) \equiv (A_2, B_2).$$

Then there exists an inertia pair (A_2, C_2) such that:

$$(A_1, B_1) \mid \equiv \mid (A_2, C_2)$$

and

$$(A_2, B_2) (\equiv) (A_2, C_2).$$

But by Theorem 176 there exists an inertia pair (A_1, C_1) such that:

$$(A_2, B_2) \mid \equiv \mid (A_1, C_1)$$

and

$$(A_1, B_1) (\equiv) (A_1, C_1).$$

Thus we may write $(A_2, B_2) \equiv (A_1, B_1)$.

Again, by Theorem 178, there is an inertia pair (B_2, D_2) such that:

$$(B_2, A_2) (\equiv) (B_2, D_2)$$

and

$$(C_2, A_2) \mid \equiv \mid (B_2, D_2).$$

Since however

$$(C_2, A_2) \mid \equiv \mid (B_1, A_1),$$

we have

$$(B_1, A_1) \mid \equiv \mid (B_2, D_2),$$

which together with the relation

$$(B_2, A_2) (\equiv) (B_2, D_2)$$

gives us

$$(B_1, A_1) \equiv (B_2, A_2).$$

If now we take instead two separation pairs (A_1, B_1) and (A_2, B_2) such that:

$$(A_1, B_1) \equiv (A_2, B_2).$$

then, by using Theorem 177 in place of Theorem 176, we may prove that:

$$(A_2, B_2) \equiv (A_1, B_1).$$

Also, by a similar method to that employed in the case of inertia pairs, but using Theorem 179 in place of Theorem 178, we may prove that:

$$(B_1, A_1) \equiv (B_2, A_2).$$

Again, if (A, B) be an inertia pair, we have

$$(A, B) | \equiv | (A, B)$$

and

$$(A, B) (\equiv) (A, B).$$

Thus we have

$$(A, B) \equiv (A, B).$$

A similar result obviously holds if (A, B) be a separation pair.

Again if (A, B) and (A, C) be inertia pairs such that:

$$(A, B) (\equiv) (A, C),$$

then since

$$(A, C) | \equiv | (A, C)$$

we may write

$$(A, B) \equiv (A, C).$$

A similar result holds if (A, B) and (A, C) be separation pairs such that:

$$(A, B) \{ \equiv \} (A, C).$$

Further it is also clear that:

$$(A_1, B_1) | \equiv | (A_2, B_2)$$

implies

$$(A_1, B_1) \equiv (A_2, B_2),$$

both when (A_1, B_1) and (A_2, B_2) are inertia pairs and when they are separation pairs.

Again if (A_1, B_1) , (A_2, B_2) and (A_3, B_3) be inertia pairs such that:

$$(A_1, B_1) \equiv (A_2, B_2)$$

and

$$(A_2, B_2) \equiv (A_3, B_3),$$

then, by the definition of congruence, there is an inertia pair (A_2, C_2) such that:

$$(A_1, B_1) | \equiv | (A_2, C_2) \dots\dots\dots(1)$$

and

$$(A_2, B_2) (\equiv) (A_2, C_2) \dots\dots\dots(2).$$

Also there is an inertia pair (A_3, C_3) such that:

$$(A_2, B_2) | \equiv | (A_3, C_3) \dots\dots\dots(3)$$

and

$$(A_3, B_3) (\equiv) (A_3, C_3) \dots\dots\dots(4).$$

Now from (2) and (3) it follows, by Theorem 176, that there is an inertia pair (A_3, D_3) such that:

$$(A_3, C_3) (\equiv) (A_3, D_3) \dots\dots\dots (5)$$

and $(A_2, C_2) | \equiv | (A_3, D_3) \dots\dots\dots (6).$

But from (1) and (6) it follows that:

$$(A_1, B_1) | \equiv | (A_3, D_3) \dots\dots\dots (7),$$

while from (4) and (5) it follows that:

$$(A_3, B_3) (\equiv) (A_3, D_3) \dots\dots\dots (8).$$

Thus from (7) and (8) it follows that:

$$(A_1, B_1) \equiv (A_3, B_3).$$

A similar result may be proved for the case of separation pairs; using Theorem 177 in place of Theorem 176.

Thus for inertia or separation pairs the general relation of congruence is a transitive relation.

Again if (A, B) be any separation pair and P be any acceleration plane containing the separation line AB , there is one single optical parallelogram in P having B as centre and A as one of its corners.

If C be the corner opposite to A then, by definition, B is the mean of A and C .

Thus by Theorem 180 we have

$$(A, B) | \equiv | (B, C).$$

But also by definition we have

$$(B, A) \{ \equiv \} (B, C).$$

And so

$$(A, B) \equiv (B, A).$$

We have not however a corresponding result in the case either of inertia or optical pairs since the elements in such pairs are asymmetrically related.

THEOREM 181.

If (O_1, D_1) and (O_2, D_2) be inertia pairs while (O_1, B_1) and (O_2, B_2) are separation pairs which are before-conjugates to (O_1, D_1) and (O_2, D_2) respectively or else after-conjugates to (O_1, D_1) and (O_2, D_2) respectively; then:

$$(1) \text{ If } (O_1, D_1) \equiv (O_2, D_2)$$

we shall also have

$$(O_1, B_1) \equiv (O_2, B_2).$$

$$(2) \text{ If } (O_1, B_1) \equiv (O_2, B_2)$$

$$\text{we shall also have } (O_1, D_1) \equiv (O_2, D_2).$$

Let us consider the first part of the theorem.

$$\text{Since } (O_1, D_1) \equiv (O_2, D_2),$$

it follows by definition that there is a pair (O_2, D') such that:

$$(O_1, D_1) | \equiv | (O_2, D')$$

$$\text{and } (O_2, D_2) (\equiv) (O_2, D').$$

Then (O_2, D') is an inertia pair and so, since (O_1, B_1) is a separation pair which is conjugate to (O_1, D_1) , it follows, by Theorem 174, that there is a separation pair, say (O_2, B') , which is conjugate to (O_2, D') and such that:

$$(O_1, B_1) | \equiv | (O_2, B').$$

Now if D_2 be *after* O_2 we shall also have D' *after* O_2 and so (O_2, D_2) and (O_2, D') will be after-conjugates to (O_2, B_2) and (O_2, B') respectively.

Thus we shall have

$$(O_2, B_2) \{ \equiv \} (O_2, B').$$

If on the other hand D_2 be *before* O_2 we shall also have D' *before* O_2 , and so (O_2, D_2) and (O_2, D') will be before-conjugates to (O_2, B_2) and (O_2, B') respectively.

Now by completing the optical parallelograms implied in the relation

$$(O_2, D_2) (\equiv) (O_2, D'),$$

we see that in this case there will be inertia pairs, say (O_2, A_2) and (O_2, A') , which will be after-conjugates to (O_2, B_2) and (O_2, B') respectively, and such that:

$$(O_2, A_2) (\equiv) (O_2, A').$$

Thus we have also in this case

$$(O_2, B_2) \{ \equiv \} (O_2, B').$$

Combining this with the relation

$$(O_1, B_1) | \equiv | (O_2, B'),$$

it follows by definition that:

$$(O_1, B_1) \equiv (O_2, B_2).$$

Thus the first part of the theorem is proved.

Consider now the second part of the theorem.

Since $(O_1, B_1) \equiv (O_2, B_2)$,

it follows by definition that there is a pair (O_2, B') such that:

$$(O_1, B_1) | \equiv | (O_2, B')$$

and $(O_2, B_2) \{ \equiv \} (O_2, B')$.

Then (O_2, B') is a separation pair and so, since (O_1, D_1) is an inertia pair which is conjugate to (O_1, B_1) , it follows, by Theorem 175, that there is an inertia pair, say (O_2, D') , which is conjugate to (O_2, B') and such that:

$$(O_1, D_1) | \equiv | (O_2, D').$$

Now if (O_1, B_1) and (O_2, B_2) be before-conjugates to (O_1, D_1) and (O_2, D_2) respectively we shall have D_1 *after* O_1 and therefore D' *after* O_2 , and also we shall have D_2 *after* O_2 .

Thus (O_2, D_2) and (O_2, D') will be after-conjugates to (O_2, B_2) and (O_2, B') respectively and so, since

$$(O_2, B_2) \{ \equiv \} (O_2, B'),$$

it follows that: $(O_2, D_2) (\equiv) (O_2, D')$.

If, on the other hand, (O_1, B_1) and (O_2, B_2) be after-conjugates to (O_1, D_1) and (O_2, D_2) respectively, we shall have D_1 *before* O_1 and therefore D' *before* O_2 and also we shall have D_2 *before* O_2 .

Thus (O_2, D_2) and (O_2, D') will be before-conjugates to (O_2, B_2) and (O_2, B') respectively.

Now by completing the optical parallelograms implied in these relations we see that there are inertia pairs, say (O_2, A_2) and (O_2, A') , which are after-conjugates to (O_2, B_2) and (O_2, B') respectively and such that D_2, O_2 and A_2 lie in one inertia line and also D', O_2 and A' lie in one inertia line.

Now since $(O_2, B_2) \{ \equiv \} (O_2, B')$,

we must have $(O_2, A_2) (\equiv) (O_2, A')$,

and therefore also in this case

$$(O_2, D_2) (\equiv) (O_2, D').$$

Combining this with the relation

$$(O_1, D_1) | \equiv | (O_2, D')$$

it follows by definition that:

$$(O_1, D_1) \equiv (O_2, D_2).$$

Thus the second part of the theorem is proved.

From Theorems 180 and 181 it follows that: if (O_1, D_1) and (O_2, D_2) be inertia pairs while (O_1, B_1) and (O_2, B_2) are separation pairs such that (O_1, B_1) is a before-conjugate to (O_1, D_1) and (O_2, B_2) is an after-conjugate to (O_2, D_2) , then:

- (1) If $(O_1, D_1) \equiv (D_2, O_2)$,
we shall also have $(O_1, B_1) \equiv (O_2, B_2)$.
- (2) If $(O_1, B_1) \equiv (O_2, B_2)$,
we shall also have $(O_1, D_1) \equiv (D_2, O_2)$.

THEOREM 182.

If (A_1, B_1) , (A_2, B_2) , (B_1, C_1) , (B_2, C_2) be pairs such that:

$$(A_1, B_1) | \equiv | (A_2, B_2)$$

and

$$(B_1, C_1) | \equiv | (B_2, C_2),$$

and if C_1 be distinct from A_1 , then we shall also have

$$(A_1, C_1) | \equiv | (A_2, C_2).$$

The elements A_1, B_1 and C_1 must lie in at least one general plane, say P_1 , and since A_2B_2 must either be parallel to A_1B_1 or identical with it, while B_2C_2 must either be parallel to B_1C_1 or identical with it, it follows that there is a general plane, say P_2 , either parallel to P_1 or identical with it which contains the elements A_2, B_2 and C_2 .

Let P' be a general plane parallel to P_1 and P_2 , and therefore distinct from both, and let (A', B') and (B', C') be pairs in P' such that:

$$(A_1, B_1) \square (A', B')$$

and

$$(B_1, C_1) \square (B', C').$$

Then, by Theorem 127, since A_1A' and C_1C' cannot lie in the same general line (owing to C_1 being distinct from A_1 and both of them lying in P_1), it follows that:

$$(A_1, C_1) \square (A', C').$$

Thus C' must be distinct from A' .

But now, since A_2, B_2 and C_2 lie in P_2 while A', B' and C' lie in the parallel general plane P' , it follows that A_2B_2 cannot be identical with $A'B'$, and B_2C_2 cannot be identical with $B'C'$.

Thus since $(A_1, B_1) | \equiv | (A_2, B_2)$

and

$$(B_1, C_1) | \equiv | (B_2, C_2),$$

it follows that we must have

$$(A_2, B_2) \sqsubset (A', B')$$

and

$$(B_2, C_2) \sqsubset (B', C').$$

Thus since C' is distinct from A' , it follows that:

$$(A_2, C_2) \sqsubset (A', C').$$

But we have seen that:

$$(A_1, C_1) \sqsubset (A', C'),$$

and so

$$(A_1, C_1) \equiv (A_2, C_2).$$

Thus the theorem is proved.

REMARKS.

One special case of this theorem deserves attention.

If B_1 be linearly between A_1 and C_1 , it follows, by Theorems 79, 95 and 119, that B' must be linearly between A' and C' and similarly B_2 must be linearly between A_2 and C_2 .

We shall require this result in proving the next theorem.

Again, since the only congruence of optical pairs is co-directional we may state the following result:

If (A_1, B_1) , (A_2, B_2) , (B_1, C_1) , (B_2, C_2) be optical pairs such that:

$$(A_1, B_1) \equiv (A_2, B_2)$$

and

$$(B_1, C_1) \equiv (B_2, C_2),$$

then if B_1 be linearly between A_1 and C_1 we shall have B_2 linearly between A_2 and C_2 and also have

$$(A_1, C_1) \equiv (A_2, C_2).$$

THEOREM 183.

If (A_1, B_1) , (A_2, B_2) , (B_1, C_1) , (B_2, C_2) be inertia pairs such that:

$$(A_1, B_1) \equiv (A_2, B_2)$$

and

$$(B_1, C_1) \equiv (B_2, C_2),$$

then if B_1 be linearly between A_1 and C_1 while B_2 is linearly between A_2 and C_2 , we shall also have

$$(A_1, C_1) \equiv (A_2, C_2).$$

If (A_2, B') be an inertia pair such that:

$$(A_1, B_1) \equiv (A_2, B'),$$

then we may take a separation line b which passes through A_2 and is normal to both A_2B_2 and A_2B' .

Let D_2 be an element in b such that (A_2, D_2) is conjugate to (A_2, B_2) and let (A_1, D_1) be a separation pair such that:

$$(A_1, D_1) \equiv (A_2, D_2).$$

Now A_1D_1 must be either parallel to A_2D_2 or identical with it, while A_1B_1 must be either parallel to A_2B' or identical with it, and so A_1D_1 must be normal to A_1B_1 .

Since A_1B_1 and A_2B_2 are inertia lines, it follows that A_1B_1 and A_1D_1 lie in an acceleration plane and also A_2B_2 and A_2D_2 lie in an acceleration plane.

Then since (A_2, D_2) is conjugate to (A_2, B_2) , it follows that D_2B_2 is an optical line.

Let B_1 be an element in A_1B_1 such that (A_1, B_1') is an after- or before-conjugate to (A_1, D_1) according as (A_2, B_2) is an after- or before-conjugate to (A_2, D_2) .

Then, by Theorem 181, since

$$(A_2, D_2) \equiv (A_1, D_1)$$

we must have

$$(A_2, B_2) \equiv (A_1, B_1').$$

But

$$(A_1, B_1) \equiv (A_2, B_2)$$

and so

$$(A_1, B_1) \equiv (A_1, B_1').$$

Thus, since A_1B_1 is an inertia line we must have B_1' identical with B_1 and so, since (A_1, B_1) is conjugate to (A_1, D_1) , it follows that D_1B_1 is an optical line.

Now let the optical line through C_1 parallel to B_1D_1 intersect A_1D_1 in F_1 and let the separation line through B_1 parallel to A_1F_1 intersect C_1F_1 in E_1 .

Then, since B_1 is linearly between A_1 and C_1 , it follows, by Theorem 78, that D_1 is linearly between A_1 and F_1 .

Let (D_2, F_2) be a pair such that:

$$(D_1, F_1) \equiv (D_2, F_2).$$

Then, by the remarks at the end of Theorem 182, D_2 will be linearly between A_2 and F_2 and we shall also have

$$(A_1, F_1) \equiv (A_2, F_2).$$

Now let the optical line through F_2 parallel to D_2B_2 intersect A_2B_2 in C_2' , and let the separation line through B_2 parallel to A_2F_2 intersect F_2C_2' in E_2 .

Then we have $(B_1, E_1) \equiv (D_1, F_1)$

and $(D_1, F_1) \equiv (D_2, F_2)$,

and therefore $(B_1, E_1) \equiv (D_2, F_2)$.

But we have also $(D_2, F_2) \equiv (B_2, E_2)$,

and so $(B_1, E_1) \equiv (B_2, E_2)$.

But now since B_1E_1 is parallel to A_1D_1 it must be normal to B_1C_1 , and since E_1C_1 is an optical line, it follows that (B_1, E_1) is conjugate to (B_1, C_1) .

Similarly, since B_2E_2 is parallel to A_2D_2 it must be normal to B_2C_2' and, since E_2C_2' is an optical line, it follows that (B_2, E_2) is conjugate to (B_2, C_2') .

But now, since D_2 is linearly between A_2 and F_2 , it follows that B_2 is linearly between A_2 and C_2' .

If then B_2 be *after* A_2 we must have C_2' *after* B_2 , while if B_2 be *before* A_2 we must have C_2' *before* B_2 .

But, since B_2 is linearly between A_2 and C_2 , it follows that if B_2 be *after* A_2 we must have C_2 *after* B_2 , while if B_2 be *before* A_2 we must have C_2 *before* B_2 .

Thus C_2' is *after* or *before* B_2 according as C_2 is *after* or *before* B_2 .

But since (B_1, C_1) and (B_2, C_2) are inertia pairs such that:

$$(B_1, C_1) \equiv (B_2, C_2),$$

it follows that C_2 is *after* or *before* B_2 according as C_1 is *after* or *before* B_1 and therefore C_2' is *after* or *before* B_2 according as C_1 is *after* or *before* B_1 .

Thus since $(B_1, E_1) \equiv (B_2, E_2)$,

it follows, by Theorem 181, that:

$$(B_1, C_1) \equiv (B_2, C_2'),$$

and since $(B_1, C_1) \equiv (B_2, C_2)$

it follows that: $(B_2, C_2) \equiv (B_2, C_2')$.

Thus since these pairs lie in the same inertia line we must have C_2' identical with C_2 .

But now C_2 will be *after* or *before* A_2 according as C_1 is *after* or *before* A_1 and so (A_2, C_2) will be an after- or before-conjugate to (A_2, F_2) according as (A_1, C_1) is an after- or before-conjugate to (A_1, F_1) .

Thus since $(A_1, F_1) \equiv (A_2, F_2)$

it follows, by Theorem 181, that:

$$(A_1, C_1) \equiv (A_2, C_2),$$

and so the theorem is proved.

THEOREM 184.

If $(A_1, B_1), (A_2, B_2), (B_1, C_1), (B_2, C_2)$ be separation pairs such that:

$$(A_1, B_1) \equiv (A_2, B_2)$$

and

$$(B_1, C_1) \equiv (B_2, C_2),$$

then if B_1 be linearly between A_1 and C_1 while B_2 is linearly between A_2 and C_2 we shall also have

$$(A_1, C_1) \equiv (A_2, C_2).$$

Let (A_1, D_1) and (A_2, D_2) be inertia pairs which are after-conjugates to (A_1, B_1) and (A_2, B_2) respectively.

Then since A_1D_1 and A_2D_2 are inertia lines it follows that A_1D_1 and A_1B_1 lie in an acceleration plane and A_2D_2 and A_2B_2 lie in an acceleration plane.

Since (A_1, D_1) is conjugate to (A_1, B_1) it follows that B_1D_1 is an optical line, and similarly B_2D_2 is an optical line.

Now let the optical line through C_1 parallel to B_1D_1 intersect A_1D_1 in F_1 , and let the optical line through C_2 parallel to B_2D_2 intersect A_2D_2 in F_2 .

Then, since B_1 is linearly between A_1 and C_1 , it follows, by Theorem 78, that D_1 is linearly between A_1 and F_1 .

Similarly D_2 is linearly between A_2 and F_2 .

Let the inertia line through B_1 parallel to A_1F_1 intersect C_1F_1 in E_1 , and let the inertia line through B_2 parallel to A_2F_2 intersect C_2F_2 in E_2 .

Then since (A_1, D_1) is an after-conjugate to (A_1, B_1) we must have D_1 *after* A_1 and since D_1 is linearly between A_1 and F_1 we must have F_1 *after* D_1 .

Thus we must have E_1 *after* B_1 and in a similar manner we can show that E_2 must be *after* B_2 .

But now, since (A_1, D_1) is conjugate to (A_1, B_1) , it follows that A_1D_1 is normal to A_1B_1 , and since B_1E_1 is parallel to A_1D_1 while A_1, B_1 and C_1 lie in one general line, it follows that B_1E_1 is normal to B_1C_1 .

Thus since C_1E_1 (that is C_1F_1) is an optical line it follows that (B_1, E_1) is conjugate to (B_1, C_1) and (A_1, F_1) is conjugate to (A_1, C_1) .

Further D_1 is *after* A_1 and F_1 is *after* D_1 and so F_1 is *after* A_1 .

Thus (B_1, E_1) and (A_1, F_1) are after-conjugates to (B_1, C_1) and (A_1, C_1) respectively.

Similarly (B_2, E_2) and (A_2, F_2) are after-conjugates to (B_2, C_2) and (A_2, C_2) respectively.

But now since $(A_1, B_1) \equiv (A_2, B_2)$,

while (A_1, D_1) and (A_2, D_2) are after-conjugates to (A_1, B_1) and (A_2, B_2) respectively, it follows, by Theorem 181, that :

$$(A_1, D_1) \equiv (A_2, D_2).$$

Similarly, since $(B_1, C_1) \equiv (B_2, C_2)$,

while (B_1, E_1) and (B_2, E_2) are after-conjugates to (B_1, C_1) and (B_2, C_2) respectively, it follows that :

$$(B_1, E_1) \equiv (B_2, E_2).$$

But we clearly have $(B_1, E_1) | \equiv | (D_1, F_1)$

and $(B_2, E_2) | \equiv | (D_2, F_2).$

Thus we have $(D_1, F_1) \equiv (D_2, F_2).$

But since D_1 is linearly between A_1 and F_1 , while D_2 is linearly between A_2 and F_2 , it follows, by Theorem 183, that

$$(A_1, F_1) \equiv (A_2, F_2).$$

We have however seen that (A_1, F_1) and (A_2, F_2) are after-conjugates to (A_1, C_1) and (A_2, C_2) respectively, and so it follows, by Theorem 181, that :

$$(A_1, C_1) \equiv (A_2, C_2).$$

Thus the theorem is proved.

THEOREM 185.

If A and B be two distinct elements and E be any element in AB distinct from A and B, while F is an element in AB such that :

$$(A, E) | \equiv | (F, B),$$

then we shall have

$$(A, F) | \equiv | (E, B).$$

Let a be a general line parallel to AB and let a general line through A intersect a in A' while parallel general lines through B and E intersect a in B' and E' respectively.

Finally let a general line through F parallel to AA' or BB' intersect a in F' .

Now we clearly have

$$(E, E') \square (B, B'),$$

and so

$$(E, E') \parallel (B, B').$$

But, since E' and A lie in parallel general lines, they must be distinct, and so, by Theorem 182,

$$(A, E') \parallel (F, B').$$

Now F cannot coincide with A for then E would require to coincide with B , contrary to hypothesis, and so FB' must be parallel to AE' .

Thus we must have

$$(A, F) \square (E', B').$$

But we also obviously have

$$(E, B) \square (E', B'),$$

and so

$$(A, F) \parallel (E, B).$$

Thus the theorem is proved.

THEOREM 186.

If B and C be two distinct elements in a separation line and O be their mean, and if A be any element in a separation line a which passes through O and is normal to BC , then:

$$(A, B) \equiv (A, C).$$

Since a is normal to BC and since they are both separation lines, it follows that a and BC lie in a separation plane, say S .

If the element A should happen to coincide with O , then since BC is a separation line, the theorem obviously holds.

Suppose next that A does not coincide with O and let d be an inertia line passing through A and normal to S .

Let P be the acceleration plane containing a and d .

Now since d is normal to S , it follows that BC is normal to d , and since BC is also normal to a , and since a and d intersect and lie in P , it follows that BC is normal to P .

Let D be the one single element common to d and the α sub-set of B .

Then BC and BD determine a general plane, say Q , which must be either an optical plane or an acceleration plane, since BD is an optical line.

But now P and Q have the general line OD in common, and since BC is normal to P it follows that BC is normal to OD .

If Q were an optical plane OD would require to be an optical line, while if Q were an acceleration plane OD would be an inertia line.

But since BD is an optical line in Q and since BD and OD intersect, it follows that Q cannot be an optical plane.

Thus Q must be an acceleration plane and OD must be an inertia line normal to the separation line BC .

Thus, since O is the mean of B and C , it follows that B , C and D are three corners of an optical parallelogram of which O is the centre.

Thus CD is an optical line.

But since AD is normal to S it must be normal to both AB and AC .

Also, since D is in the α sub-set of B and is distinct from B it follows that D is *after* B .

Thus, since AB is a separation line while AD is an inertia line, it follows that D is *after* A , and accordingly (A, D) is an after-conjugate to both (A, B) and (A, C) .

Thus we have $(A, B) \equiv (A, C)$,

and so the theorem is proved.

THEOREM 187.

If A , B and C be three distinct elements which lie in a separation plane S , but do not all lie in one general line, and if O be the mean of B and C while

$$(A, B) \equiv (A, C),$$

then AO is normal to BC .

Let d be an inertia line passing through A and normal to S and let P be the acceleration plane containing d and AO .

Then d is normal to both AB and AC and, since

$$(A, B) \equiv (A, C),$$

there is one definite element, say D , in d such that (A, D) is an after-conjugate to both (A, B) and (A, C) .

Thus BD and CD are optical lines and since they intersect they must lie in an acceleration plane, say Q .

But now since O is the mean of B and C , it follows that B , C and D are three corners of an optical parallelogram whose centre is O , and therefore BC is normal to OD .

But OD is common to both Q and P , while BC (since it lies in S) is normal to AD , which also lies in P .

Thus BC is normal to two intersecting general lines in P and therefore BC is normal to P .

But AO lies in P and therefore AO is normal to BC .

Thus the theorem is proved.

THEOREM 188.

If A , B and C be three distinct elements which lie in a separation plane S , but do not all lie in one general line, and if

$$(A, B) \equiv (A, C),$$

and if O be an element in BC such that AO is normal to BC , then O is the mean of B and C .

Let O' be the mean of B and C .

Then, by Theorem 187, AO' is normal to BC , and, by hypothesis, AO is normal to BC .

But both AO' and AO pass through the element A and lie in the separation plane S and we have already seen that there is only one general line in a given separation plane which passes through a given element and is normal to another general line in the separation plane.

Thus AO must be identical with AO' and therefore O must be identical with O' .

It follows that O is the mean of B and C and so the theorem is proved.

Definitions. If A and B be two distinct elements, then the set of all elements lying linearly between A and B will be called the *segment* AB .

The elements A and B will be called the *ends* of the segment, but are not included in it.

The set of elements obtained by including the ends will be called a *linear interval*.

If A and B be two distinct elements, then the set of elements such as X where B is linearly between A and X may be called the *prolongation of the segment* AB *beyond* B .

Such a set of elements will also be spoken of as a *general half-line*.

The element B will be called the *end* of the general half-line.

We shall describe segments and general half-lines as *optical*, *inertia*, or *separation*, according as they lie in optical, inertia, or separation lines.

It is easy to see that any element B in a general line divides the remaining elements of the general line into two sets such that B is linearly between any two elements of opposite sets, but is not linearly between any two elements of the same set.

For let A and C be two elements in the general line such that B is linearly between A and C , and let A' be another element in it distinct from A and such that B is linearly between A' and C .

Then by the analogue of Peano's Axiom 10 since A and A' are distinct we must either have A' linearly between B and A or A linearly between B and A' .

Thus we cannot have B linearly between A and A' .

Again if C' be an element distinct from C and such that B is linearly between A and C' it follows similarly that we must have either C' linearly between B and C or C linearly between B and C' and cannot have B linearly between C and C' .

If now A' be linearly between A and B then since B is linearly between A and C' it follows, by the analogue of Peano's Axiom 8, that A' is linearly between A and C' .

But now since B is linearly between A and C' and also A' is linearly between A and C' it follows, by the analogue of Peano's Axiom 9, that either B is linearly between A and A' or B is identical with A' or B is linearly between A' and C' .

The first and second of these alternatives are impossible since by hypothesis A' is linearly between A and B .

Thus we must have B linearly between A' and C' .

Next take the case where A is linearly between A' and B .

Since we also have B linearly between A and C' , it follows by the analogue of Peano's Axiom 11 that B is linearly between A' and C' .

This shows that B divides the remaining elements of the general line AC into two sets in the manner above stated.

It is clear that these sets are general half-lines.

If elements X and Y lie in the same general half-line whose end is B , they will be said to lie *on the same side of B* .

If, on the other hand, B be linearly between X and Y , then these elements will be said to lie *on opposite sides of B* .

A general half-line whose end is B and which contains an element X may be denoted by *general half-line BX* .

It is also easy to see that any general line b in a general plane P divides the remaining elements of P into two sets such that if A and C be any two elements of opposite sets then b will intersect AC in an element linearly between A and C ; while if A and A' be two elements of the same set, then b will not intersect AA' in any element linearly between A and A' .

For let A and C be two elements in P and let them be such that b intersects AC in an element B linearly between A and C .

Also let A' be another element in P distinct from A and such that b intersects $A'C$ in an element B' linearly between A' and C .

If A' should happen to lie in AC then B' will coincide with B , and since b cannot intersect AC in any other element besides B , it follows that b does not intersect AA' in an element linearly between A and A' .

If A' does not lie in AC then it follows from Theorem 128 (2) that b cannot intersect AA' in an element linearly between A and A' .

Again if C' be an element lying in P and distinct from C and such that b intersects AC' in an element B'' linearly between A and C' , it follows in a similar manner that b cannot intersect CC' in an element linearly between C and C' .

Now C' either does or does not lie in $A'A$; if C' does lie in $A'A$ then since B'' is linearly between A and C' , but is not linearly between A and A' , it follows that B'' must be linearly between A' and C' .

If, on the other hand, C' does not lie in $A'A$, then since B'' is linearly between A and C' , and since b does not coincide with either AC' or $B''A'$, it follows by Theorem 128 (1) that b either intersects $A'A$ in an element linearly between A' and A , or else intersects $A'C'$ in an element linearly between A' and C' .

But the first of these alternatives is excluded and so the second must hold.

This shows that the general plane P is divided by the general line b in the manner above stated.

If elements X and Y lie in the general plane P , but not in the general line b , they will be said to lie *on the same side* of b if they both lie in the same set and will be said to lie *on opposite sides* of b if X lies in one of the sets and Y in the other set.

Definition. If a general line b lies in a general plane P then either of the sets of elements on one side of b will be called a *general half-plane*.

The general line b will be called the *boundary* of the general half-plane.

The following important result which may be conveniently expressed in the nomenclature of general half-lines can be easily proved.

If (A_1, B_1) , (A_2, B_2) , (A_1, C_1) , (A_2, C_2) be inertia, optical or separation pairs such that:

$$(A_1, B_1) \equiv (A_2, B_2),$$

and

$$(A_1, C_1) \equiv (A_2, C_2),$$

then if B_1 be linearly between A_1 and C_1 and if C_2 lies in the general half-line A_2B_2 , we shall also have B_2 linearly between A_2 and C_2 .

In the case of optical pairs the above congruences imply that A_1B_1 and A_2B_2 are the same or parallel optical lines, but nothing of this sort is implied in the case of inertia or separation pairs.

In all cases there is an element, say C_2' , in A_2B_2 and on the opposite side of B_2 to that on which A_2 lies and such that:

$$(B_1, C_1) \equiv (B_2, C_2').$$

Then in all cases it follows that:

$$(A_1, C_1) \equiv (A_2, C_2'),$$

and so

$$(A_2, C_2) \equiv (A_2, C_2').$$

But C_2 and C_2' both lie in A_2B_2 and on the same side of A_2 , and must therefore be identical.

Thus B_2 is linearly between A_2 and C_2 .

Definition. If A, B, C be three distinct elements which do not all lie in one general line, then the three segments AB, BC, CA , together with the three elements A, B, C , will be called a *general triangle*, or briefly a *triangle* in an acceleration, optical, or separation plane, as the case may be.

The elements A, B, C will be called the *corners* while the segments AB, BC, CA will be called the *sides* of the general triangle.

THEOREM 189.

If A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1 and A_2, B_2, C_2 be the corners of a triangle in a separation plane P_2 and if further

$$(C_1, A_1) \equiv (C_2, A_2),$$

$$(C_1, B_1) \equiv (C_2, B_2),$$

while B_1C_1 is normal to A_1C_1 , and B_2C_2 is normal to A_2C_2 , then we shall also have

$$(A_1, B_1) \equiv (A_2, B_2).$$

In order to prove this theorem we shall consider a number of special cases on which the general proof is made to depend.

CASE I. B_2 identical with B_1 and C_2 identical with C_1 , while P_2 is identical with P_1 .

In this case since the separation lines A_1C_1 and A_2C_1 are both normal to B_1C_1 and both lie in the separation plane P_1 and pass through the element C_1 , they must be identical.

If further A_2 should coincide with A_1 the result is obvious, and so we shall suppose that A_2 does not coincide with A_1 .

Now since (C_1, A_1) , &c. lie in the separation plane P_1 they must all be separation pairs and since in this case

$$(C_1, A_2) \equiv (A_2, C_1),$$

it follows that:

$$(A_2, C_1) \equiv (C_1, A_1).$$

Thus C_1 must be the mean of A_1 and A_2 and therefore, by Theorem 186, we have

$$(B_1, A_1) \equiv (B_1, A_2),$$

or

$$(A_1, B_1) \equiv (A_2, B_2).$$

CASE II. B_2 identical with B_1 and C_2 identical with C_1 , while P_1 and P_2 lie in the same separation threefold W .

If P_2 should be identical with P_1 this case reduces to Case I, and so we shall suppose them distinct.

Now since A_1 , B_1 and A_2 are three distinct elements in W which do not lie in one general line, it follows that A_1 , B_1 and A_2 lie in a separation plane, say R , which must be distinct from both P_1 and P_2 , since these latter two separation planes are supposed distinct.

Similarly A_1 , C_1 and A_2 lie in a separation plane, say S , which is also distinct from P_1 and P_2 .

Now let O be the mean of A_1 and A_2 .

Then, by Theorem 187, since

$$(C_1, A_1) \equiv (C_1, A_2),$$

it follows that C_1O is normal to A_1A_2 .

But since B_1C_1 is normal to A_1C_1 and to A_2C_1 which are distinct intersecting separation lines, it follows that B_1C_1 is normal to S and therefore must be normal to A_1A_2 .

Thus A_1A_2 is normal to the two intersecting separation lines B_1C_1 and C_1O and must therefore be normal to every general line in the general plane containing them.

It follows that A_1A_2 is normal to B_1O .

But now, by Theorem 186, since O is the mean of A_1 and A_2 and since B_1 , A_1 , A_2 lie in a separation plane, it follows that:

$$(B_1, A_1) \equiv (B_1, A_2),$$

or

$$(A_1, B_1) \equiv (A_2, B_2).$$

CASE III. C_2 identical with C_1 and P_2 identical with P_1 .

Let b be a separation line passing through C_1 and normal to P_1 and let B' be an element in b such that:

$$(C_1, B') \equiv (C_1, B_1).$$

Then we shall also have

$$(C_1, B') \equiv (C_1, B_2).$$

Now the separation plane P and the separation line b determine a separation threefold, say W , which contains A_1 , B_1 , C_1 , A_2 , B_2 , B' .

Again, since $B'C_1$ is normal to P , it must be normal to C_1A_1 , C_1A_2 , C_1B_1 and C_1B_2 .

Then since $(C_1, B_1) \equiv (C_1, B')$,

it follows by Case II that:

$$(A_1, B_1) \equiv (A_1, B') \dots \dots \dots (1).$$

Again since $(C_1, A_1) \equiv (C_1, A_2)$,

it follows by Case II that:

$$(A_1, B') \equiv (A_2, B') \dots \dots \dots (2).$$

Further, by Case II, since

$$(C_1, B') \equiv (C_1, B_2),$$

it follows that:

$$(A_2, B') \equiv (A_2, B_2) \dots \dots \dots (3).$$

Thus from (1), (2) and (3) it follows that:

$$(A_1, B_1) \equiv (A_2, B_2).$$

CASE IV. P_2 either identical with P_1 or parallel to P_1 .

There is, as we have already seen, one single element, say A' , such that:

$$(C_1, A_1) \mid \equiv \mid (C_2, A').$$

Similarly there is one single element, say B' , such that:

$$(C_1, B_1) \mid \equiv \mid (C_2, B').$$

Now, since P_2 is either identical with P_1 or parallel to P_1 , it follows that P_2 must contain C_2A' and C_2B' .

Also since C_2A' must be either parallel to C_1A_1 or identical with it, and since C_2B' must be either parallel to C_1B_1 or identical with it, then since B_1C_1 is normal to A_1C_1 , it follows that $B'C_2$ is normal to $A'C_2$.

But now, by Theorem 182, we must have

$$(A_1, B_1) \mid \equiv \mid (A', B').$$

Also since $(C_1, A_1) \equiv (C_2, A_2)$,

it follows that: $(C_2, A_2) \equiv (C_2, A')$,

and since $(C_1, B_1) \equiv (C_2, B_2)$,

it follows that: $(C_2, B_2) \equiv (C_2, B')$.

Thus, by Case III, it follows that:

$$(A', B') \equiv (A_2, B_2).$$

Since however we have

$$(A_1, B_1) \equiv (A', B'),$$

it follows that: $(A_1, B_1) \equiv (A_2, B_2)$.

CASE V. P_1 and P_2 lie in the same separation threefold W .

If P_1 and P_2 have no element in common, then since they both lie in W they must be parallel to one another and the result follows from Case IV.

We shall therefore suppose that P_1 and P_2 have an element in common, but are distinct.

Then, by Theorem 152, they have a second element in common, and therefore have a general line in common which we shall call b .

Let C be any element in b and let a_1 and a_2 be separation lines passing through C and normal to b and lying in P_1 and P_2 respectively.

Let B be an element in b such that:

$$(C_1, B_1) \equiv (C, B).$$

Then we shall also have

$$(C_2, B_2) \equiv (C, B).$$

Let A_1' and A_2' be elements in a_1 and a_2 respectively such that:

$$(C_1, A_1) \equiv (C, A_1'),$$

and

$$(C_2, A_2) \equiv (C, A_2').$$

Then since

$$(C_1, A_1) \equiv (C_2, A_2),$$

we have

$$(C, A_1') \equiv (C, A_2').$$

Thus by Case II it follows that:

$$(A_1', B) \equiv (A_2', B) \dots\dots\dots(1).$$

But by Case IV it follows that:

$$(A_1, B_1) \equiv (A_1', B) \dots\dots\dots(2),$$

and similarly it follows that:

$$(A_2, B_2) \equiv (A_2', B) \dots\dots\dots(3).$$

Thus from (1), (2) and (3) it follows that:

$$(A_1, B_1) \equiv (A_2, B_2).$$

Thus whether P_1 and P_2 are identical, or parallel, or whether they have a general line in common, the theorem holds provided P_1 and P_2 lie in the same separation threefold W .

CASE VI. P_1 and P_2 do not lie in one separation threefold.

In this case we may take one separation threefold, say W_1 , which contains P_1 and another separation threefold, say W_2 , which contains P_2 .

Now W_2 may be either parallel to W_1 , or else not parallel to it.

If X be any element in W_2 but not in W_1 then, if W_2 be not parallel to W_1 , there must be at least one separation line passing through X and lying in W_2 which is not parallel to any separation line in W_1 and which therefore, by Theorem 164, must intersect W_1 .

Thus if W_2 be not parallel to W_1 it must have an element in common with it and so, by Theorem 166, W_1 and W_2 must have a general plane in common.

Suppose then first that W_2 is parallel to W_1 and let C be any element in W_2 .

Then there is a general line, say a , passing through C and lying in W_2 which is parallel to C_1A_1 .

Similarly there is a general line, say b , passing through C and lying in W_2 which is parallel to C_1B_1 .

Thus since B_1C_1 is normal to A_1C_1 , it follows that b is normal to a .

Now let A and B be elements in a and b respectively such that :

$$(C_1, A_1) \equiv (C, A)$$

and

$$(C_1, B_1) \equiv (C, B).$$

Then by Theorem 182 we must have

$$(A_1, B_1) \equiv (A, B).$$

But now since

$$(C_1, A_1) \equiv (C_2, A_2),$$

it follows that :

$$(C_2, A_2) \equiv (C, A),$$

and since

$$(C_1, B_1) \equiv (C_2, B_2),$$

it follows that :

$$(C_2, B_2) \equiv (C, B).$$

Thus since A, B and C lie in W_2 which also contains P_2 , it follows by Case V that :

$$(A, B) \equiv (A_2, B_2).$$

Thus since

$$(A_1, B_1) \equiv (A, B),$$

it follows that :

$$(A_1, B_1) \equiv (A_2, B_2).$$

Suppose next that W_2 is not parallel to W_1 and let S be the general plane which they have in common and which must be a separation plane.

Let a be any separation line in S , and C be any element in it.

Let b be the separation line which passes through C and lies in S and which is normal to a .

Let A and B be elements in a and b respectively such that :

$$(C_1, A_1) \equiv (C, A)$$

and

$$(C_1, B_1) \equiv (C, B).$$

Then we shall also have

$$(C_2, A_2) \equiv (C, A)$$

and

$$(C_2, B_2) \equiv (C, B).$$

But since P_1 and S lie in W_1 , it follows by Case V that :

$$(A_1, B_1) \equiv (A, B).$$

Also since P_2 and S lie in W_2 , it follows by Case V that :

$$(A_2, B_2) \equiv (A, B).$$

Thus we get finally. $(A_1, B_1) \equiv (A_2, B_2)$.

Combining now Cases V and VI we see that whether P_1 and P_2 lie in one separation threefold or not, the theorem still holds.

Thus the theorem holds in all cases.

THEOREM 190.

If A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1 and A_2, B_2, C_2 be the corners of a triangle in a separation plane P_2 , and if further

$$(C_1, A_1) \equiv (C_2, A_2),$$

$$(A_1, B_1) \equiv (A_2, B_2),$$

while B_1C_1 is normal to A_1C_1 , and B_2C_2 is normal to A_2C_2 , then we shall also have

$$(C_1, B_1) \equiv (C_2, B_2).$$

Let B_2' be an element in B_2C_2 and on the same side of C_2 as is B_2 and such that :

$$(C_1, B_1) \equiv (C_2, B_2').$$

Then by Theorem 189 we must have

$$(A_1, B_1) \equiv (A_2, B_2'),$$

and so we must have $(A_2, B_2) \equiv (A_2, B_2')$.

Suppose now, if possible, that B_2' is distinct from B_2 and let O be the mean of B_2 and B_2' .

Then, by Theorem 187, A_2O must be normal to B_2C_2 .

But A_2C_2 is normal to B_2C_2 and so since P_2 is a separation plane we must have A_2O identical with A_2C_2 and therefore O would be identical with C_2 .

Since however O is supposed to be the mean of B_2 and B_2' it would require to be linearly between them and so B_2 and B_2' would be on opposite sides of C_2 , contrary to hypothesis.

Thus the supposition that B_2 and B_2' are distinct leads to a contradiction and so B_2' must be identical with B_2 .

But $(C_1, B_1) \equiv (C_2, B_2')$,
and therefore $(C_1, B_1) \equiv (C_2, B_2)$
as was to be proved.

THEOREM 191.

If A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1 and A_2, B_2, C_2 be the corners of a triangle in a separation plane P_2 and if further

$$(A_1, B_1) \equiv (A_2, B_2),$$

$$(A_1, C_1) \equiv (A_2, C_2),$$

$$(B_1, C_1) \equiv (B_2, C_2),$$

while A_1C_1 is normal to B_1C_1 , then we shall also have A_2C_2 normal to B_2C_2 .

Let a be a separation line passing through C_2 and lying in P_2 and which is normal to B_2C_2 and let A_2' be an element in a on the same side of B_2C_2 as is A_2 and such that :

$$(A_2', C_2) \equiv (A_1, C_1).$$

Then, by Theorem 189, we shall have

$$(A_2', B_2) \equiv (A_1, B_1).$$

It follows that we must have

$$(A_2, B_2) \equiv (A_2', B_2),$$

and further

$$(A_2, C_2) \equiv (A_2', C_2).$$

Now if A_2' lies in A_2B_2 it must be identical with A_2 for there is only one element, say A , distinct from A_2 and lying in A_2B_2 and such that :

$$(A_2, B_2) \equiv (A, B_2),$$

and this element A lies on the opposite side of B_2 to that on which A_2 lies.

Thus since A_2 and A_2' lie on the same side of B_2C_2 and therefore on the same side of B_2 , it follows that A_2' must be identical with A_2 .

Similarly if A_2' lies in A_2C_2 it must be identical with A_2 .

Suppose now, if possible, that A_2' is distinct from A_2 and lies neither in A_2B_2 nor in A_2C_2 and let O be the mean of A_2 and A_2' .

Then, by Theorem 187, B_2O must be normal to A_2A_2' and similarly C_2O must be normal to A_2A_2' .

Thus since B_2O and C_2O lie in the same separation plane as A_2A_2' it follows that B_2O and C_2O must be the same general line which accordingly must be identical with B_2C_2 .

Thus O would require to lie in B_2C_2 .

But since O is supposed to be the mean of A_2 and A_2' it must be linearly between them and so A_2 and A_2' would be on opposite sides of B_2C_2 , contrary to hypothesis.

Thus the assumption that A_2' is distinct from A_2 leads to a contradiction and so A_2' is identical with A_2 .

But $A_2'C_2$ is normal to B_2C_2 by hypothesis and so A_2C_2 is normal to B_2C_2 as was to be proved.

THEOREM 192.

If A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1 and A_2, B_2, C_2 be the corners of a triangle in a separation plane P_2 , and if further

$$\begin{aligned}(A_1, B_1) &\equiv (A_2, B_2), \\ (A_1, C_1) &\equiv (A_2, C_2), \\ (B_1, C_1) &\equiv (B_2, C_2),\end{aligned}$$

while N_1 is an element in B_1C_1 such that A_1N_1 is normal to B_1C_1 , and N_2 is an element in B_2C_2 such that A_2N_2 is normal to B_2C_2 ; and if N_1 be distinct from both B_1 and C_1 , then N_2 will be distinct from both B_2 and C_2 , and we shall also have

$$\begin{aligned}(A_1, N_1) &\equiv (A_2, N_2), \\ (B_1, N_1) &\equiv (B_2, N_2), \\ (C_1, N_1) &\equiv (C_2, N_2).\end{aligned}$$

If N_1 be linearly between B_1 and C_1 let N_2' be an element in B_2C_2 and on the same side of B_2 as is C_2 and such that:

$$(B_1, N_1) \equiv (B_2, N_2').$$

Let C_2' be an element in B_2C_2 and on the opposite side of N_2' to that on which B_2 lies and such that:

$$(N_1, C_1) \equiv (N_2', C_2').$$

Then, by Theorem 184, we shall have

$$(B_1, C_1) \equiv (B_2, C_2'),$$

and so

$$(B_2, C_2) \equiv (B_2, C_2').$$

But C_2 and C_2' both lie on the same side of B_2 and so they must be identical.

Thus we must have

$$(C_1, N_1) \equiv (C_2, N_2').$$

Again if C_1 be linearly between B_1 and N_1 , let N_2' be an element in B_2C_2 and on the opposite side of C_2 to that on which B_2 lies and such that:

$$(C_1, N_1) \equiv (C_2, N_2').$$

Then, by Theorem 184, we shall have

$$(B_1, N_1) \equiv (B_2, N_2').$$

Similarly if B_1 be linearly between C_1 and N_1 , let N_2' be an element in C_2B_2 and on the opposite side of B_2 to that on which C_2 lies and such that:

$$(B_1, N_1) \equiv (B_2, N_2').$$

Then, by Theorem 184, we shall have

$$(C_1, N_1) \equiv (C_2, N_2').$$

Thus in all three cases N_2' has been taken in B_2C_2 in such a manner that:

$$(B_1, N_1) \equiv (B_2, N_2')$$

and

$$(C_1, N_1) \equiv (C_2, N_2').$$

Now let a be a separation line lying in P_2 and passing through N_2' and normal to B_2C_2 .

Let an element A_2' be selected in a and on the same side of B_2C_2 as A_2 lies and such that:

$$(N_1, A_1) \equiv (N_2', A_2').$$

Then, by Theorem 189, it follows that:

$$(A_1, B_1) \equiv (A_2', B_2)$$

and

$$(A_1, C_1) \equiv (A_2', C_2).$$

Thus we must have

$$(A_2, B_2) \equiv (A_2', B_2)$$

and

$$(A_2, C_2) \equiv (A_2', C_2).$$

Then as in the last theorem we may prove that the elements A_2 and A_2' must be identical and, since A_2N_2' is normal to B_2C_2 and intersects it in the element N_2' and since P_2 is a separation plane, it follows that N_2' is identical with N_2 , and therefore N_2 is distinct from both B_2 and C_2 .

Thus we must have

$$(A_1, N_1) \equiv (A_2, N_2),$$

$$(B_1, N_1) \equiv (B_2, N_2),$$

$$(C_1, N_1) \equiv (C_2, N_2),$$

and so the theorem is proved.

It is also evident from the manner in which N_2' was determined that we must have

| | |
|--------------|---|
| | N_2 linearly between B_2 and C_2 , |
| or | B_2 linearly between N_2 and C_2 , |
| or | C_2 linearly between N_2 and B_2 , |
| according as | N_1 is linearly between B_1 and C_1 , |
| or | B_1 is linearly between N_1 and C_1 , |
| or | C_1 is linearly between N_1 and B_1 . |

THEOREM 193.

If A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1 , and A_2, B_2, C_2 be the corners of a triangle in a separation plane P_2 , and if D_1 be an element in B_1C_1 such that C_1 is linearly between B_1 and D_1 , while D_2 is an element in B_2C_2 such that C_2 is linearly between B_2 and D_2 ; and if further

$$(A_1, B_1) \equiv (A_2, B_2),$$

$$(B_1, C_1) \equiv (B_2, C_2),$$

$$(C_1, A_1) \equiv (C_2, A_2),$$

$$(B_1, D_1) \equiv (B_2, D_2),$$

then we shall also have $(A_1, D_1) \equiv (A_2, D_2)$.

It is easy to see that:

$$(C_1, D_1) \equiv (C_2, D_2),$$

for let D_2' be an element in B_2C_2 and on the opposite side of C_2 to that on which B_2 lies and such that:

$$(C_1, D_1) \equiv (C_2, D_2').$$

Then, by Theorem 184, we must have

$$(B_1, D_1) \equiv (B_2, D_2').$$

Thus

$$(B_2, D_2) \equiv (B_2, D_2'),$$

and, since D_2 and D_2' lie on the same side of B_2 , it follows that they are identical.

Thus we have $(C_1, D_1) \equiv (C_2, D_2)$.

Now if A_1B_1 should happen to be normal to B_1C_1 , it follows by Theorem 191 that we must also have A_2B_2 normal to B_2C_2 .

Thus, by Theorem 189, we must have

$$(A_1, D_1) \equiv (A_2, D_2).$$

Again if C_1A_1 should happen to be normal to B_1C_1 it follows, by Theorem 191, that C_2A_2 is normal to B_2C_2 and so, by Theorem 189, we must again have

$$(A_1, D_1) \equiv (A_2, D_2).$$

Next consider the cases where neither A_1B_1 nor C_1A_1 are normal to B_1C_1 .

Let a separation line passing through A_1 and lying in P_1 , and which is normal to B_1C_1 , intersect B_1C_1 in N_1 .

Then N_1 is distinct from both B_1 and C_1 and so if N_2 be an element in B_2C_2 such that A_2N_2 is normal to B_2C_2 , it follows, by Theorem 192, that :

$$(A_1, N_1) \equiv (A_2, N_2),$$

$$(B_1, N_1) \equiv (B_2, N_2),$$

$$(C_1, N_1) \equiv (C_2, N_2).$$

If now N_1 should happen to coincide with D_1 we should have C_1 linearly between N_1 and B_1 and therefore C_2 linearly between N_2 and B_2 and so since then we should have

$$(C_2, N_2) \equiv (C_2, D_2),$$

and N_2 and D_2 would lie on the same side of C_2 , it would follow that N_2 must coincide with D_2 and therefore again

$$(A_1, D_1) \equiv (A_2, D_2).$$

Next suppose that N_1 does not coincide with D_1 .

Then if N_1 lies on the opposite side of B_1 to that on which C_1 lies, N_2 will lie on the opposite side of B_2 to that on which C_2 lies; while if N_1 lies on the same side of B_1 as that on which C_1 lies, N_2 will lie on the same side of B_2 as that on which C_2 lies.

Suppose first that N_1 and C_1 lie on opposite sides of B_1 .

Then N_1 and D_1 will also lie on opposite sides of B_1 while N_2 and D_2 will lie on opposite sides of B_2 .

Thus since $(B_1, N_1) \equiv (B_2, N_2)$

and $(B_1, D_1) \equiv (B_2, D_2),$

it follows, by Theorem 184, that :

$$(N_1, D_1) \equiv (N_2, D_2).$$

If N_1 and C_1 lie on the same side of B_1 we shall also have N_1 and D_1 on the same side of B_1 , while N_2 and D_2 will be on the same side of B_2 .

Consider first the case where N_1 is linearly between B_1 and D_1 and let \bar{D}_2 be an element in B_2N_2 and on the opposite side of N_2 to that on which B_2 lies and such that:

$$(N_2, \bar{D}_2) \equiv (N_1, D_1).$$

Then, by Theorem 184, we must have

$$(B_1, D_1) \equiv (B_2, \bar{D}_2).$$

But since

$$(B_1, D_1) \equiv (B_2, D_2),$$

it follows that:

$$(B_2, D_2) \equiv (B_2, \bar{D}_2).$$

Thus since D_2 and \bar{D}_2 lie on the same side of B_2 they must be identical, and so

$$(N_1, D_1) \equiv (N_2, D_2).$$

In the case where D_1 is linearly between B_1 and N_1 we may use a similar method to prove the same result.

Thus in all cases where N_1 does not coincide with D_1 , B_1 or C_1 we have

$$(N_1, D_1) \equiv (N_2, D_2)$$

and

$$(A_1, N_1) \equiv (A_2, N_2).$$

Thus, by Theorem 189, it follows that:

$$(A_1, D_1) \equiv (A_2, D_2),$$

and so the theorem holds in all cases.

Definitions. If O and X_0 be two distinct elements in a separation plane S , then the set of all elements in S such as X where

$$(O, X) \equiv (O, X_0)$$

will be called a *separation circle*.

The element O will be called the *centre* of the separation circle.

Any one of the linear intervals such as OX will be called a *radius* of the separation circle.

If X_1 and X_2 be two elements of the separation circle such that X_1X_2 passes through O , then the linear interval X_1X_2 will be called a *diameter* of the separation circle.

Any element which lies in a radius but which is not an element of the separation circle itself will be said to lie *inside* or in the *interior* of the separation circle.

Any element which lies in S but not in a radius will be said to lie *outside* or *exterior* to the separation circle.

THEOREM 194.

If a separation circle and a separation line both lie in a separation plane S , they cannot have more than two elements in common.

Let a be the separation line and O the centre of the separation circle which we shall suppose to have at least one element A in common with a .

Then a either does or does not pass through O .

If a passes through O then we know that there is one single element, say B , lying in a and distinct from A and such that :

$$(O, A) \equiv (O, B).$$

Thus B is a second element common to the separation circle and the separation line and in this case no other such element exists.

Next suppose that a does not pass through O .

Two cases are possible :

- (1) OA is normal to a ,
- (2) OA is not normal to a .

In Case (1) suppose, if possible, that B is a second element common to the separation circle and separation line and let M be the mean of A and B .

Then we should have $(O, A) \equiv (O, B)$,

and since A , B and O would not lie in one general line, it would follow, by Theorem 187, that OM must be normal to AB .

But by hypothesis OA is normal to AB , and, since S is a separation plane, we cannot have a second separation line passing through O and lying in S and which is normal to AB .

Thus no such element as B exists and so in this case the separation circle and the separation line have only one element in common.

Consider next Case (2) where OA is not normal to a , and let N be an element in a such that ON is normal to a .

Let B be an element in a and on the opposite side of N to that on which A lies and such that :

$$(N, A) \equiv (N, B).$$

Then N will be the mean of A and B , and so, by Theorem 186,

$$(O, A) \equiv (O, B).$$

Thus B is a second element common to the separation circle and the separation line.

Suppose now, if possible, that B' is another element in a distinct from A and B and which is an element of the separation circle.

Then we should have

$$(O, A) \equiv (O, B')$$

and

$$(O, B) \equiv (O, B').$$

Thus since ON is normal to AB' it would follow, by Theorem 188, that N must be the mean of A and B' and of B and B' , in addition to being the mean of A and B , which is clearly impossible, since in any acceleration plane containing a there is only one optical parallelogram having N as centre and A as one of its corners.

Thus no such element as B' can exist and so the separation circle and separation line cannot in any case have more than two elements in common.

THEOREM 195.

If a separation circle in a separation plane S pass through an element A which is inside, and another element B which is outside a second separa-

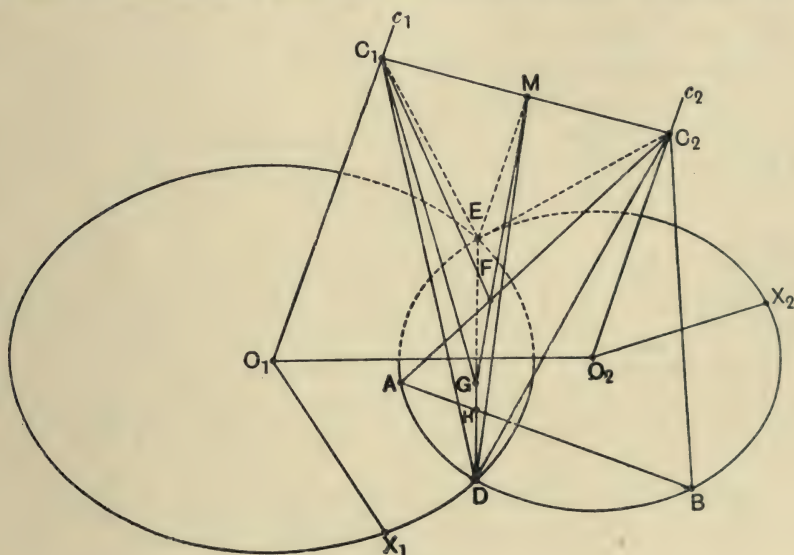


Fig. 43.

tion circle in S , then the two separation circles have two elements in common which lie on opposite sides of the separation line AB .

Let O_1 be the centre of the separation circle inside which A lies and outside which B lies, and let O_2 be the centre of the other separation circle.

Let W be a rotation threefold containing S .

Then, by Theorem 160, through any element of S there is one and only one general line which lies in W and is normal to S .

Further such a general line must be an inertia line, since otherwise W would be an optical or a separation threefold.

Let c_1 and c_2 be inertia lines passing through O_1 and O_2 respectively and which lie in W and are normal to S .

Then c_1 and c_2 must be either parallel or identical, but we shall show in the course of our proof that they cannot be identical.

Let X_1 and X_2 be any elements of the separation circles whose centres are O_1 and O_2 respectively and let elements C_1 and C_2 be taken in c_1 and c_2 respectively such that (O_1, C_1) is an after-conjugate to (O_1, X_1) , and (O_2, C_2) an after-conjugate to (O_2, X_2) .

Then if X_1' be any element of the separation circle whose centre is O_1 we shall have

$$(O_1, X_1) \equiv (O_1, X_1'),$$

and so, since O_1C_1 is normal to O_1X_1' , it follows that (O_1, C_1) is also an after-conjugate to (O_1, X_1') and therefore $X_1'C_1$ is an optical line.

Similarly if X_2' be any element of the separation circle whose centre is O_2 , then $X_2'C_2$ will be an optical line.

Thus AC_2 and BC_2 are optical lines.

Now if we take a separation line in S which passes through O_1 and A there will be two elements in it, say Y and Y' , such that:

$$(O_1, Y) \equiv (O_1, X_1)$$

and

$$(O_1, Y') \equiv (O_1, X_1).$$

Then since A lies inside the separation circle whose centre is O_1 it must lie in one of the radii O_1Y or O_1Y' and be distinct from Y and Y' .

Thus A must lie linearly between Y and Y' .

But since (O_1, C_1) is an after-conjugate to (O_1, Y) and (O_1, Y') it follows that C_1 is *after* both Y and Y' and so, by Theorem 129 (b), C_1A is an inertia line and A is *before* C_1 .

Again if we take a separation line in S which passes through O_1 and B there will be two elements in it, say Z and Z' , such that:

$$(O_1, Z) \equiv (O_1, X_1)$$

and

$$(O_1, Z') \equiv (O_1, X_1),$$

and since B lies outside the separation circle whose centre is O_1 , it does not lie in either of the radii O_1Z or O_1Z' .

Thus B does not lie linearly between Z and Z' and does not coincide with either Z or Z' .

It follows that we must either have Z linearly between Z' and B , or else Z' linearly between Z and B .

Now it is evident (as in the case of Y and Y') we must have C_1 *after* both Z and Z' .

But B cannot be *after* C_1 for then B would be *after* Z which is impossible, since ZB lies in S and is therefore a separation line.

Further C_1 cannot be *after* B for then, by Theorem 129 (b), either C_1Z or C_1Z' would require to be an inertia line, whereas we know that they are both optical lines.

Thus B is neither *before* nor *after* C_1 and so C_1B is a separation line.

Now C_2 cannot be identical with or *before* C_1 , for, since B must be *before* C_2 , it would follow that B was *before* C_1 , which we have already seen is not so.

Neither can C_2 be *after* C_1 , for we have seen that C_1 is *after* A and that C_1A is an inertia line, and so we should have C_1 *after* one element of the optical line AC_2 and *before* another element of it and yet not itself an element of it, which is impossible by Theorem 12.

Thus C_2 is neither *before* nor *after* C_1 , and is distinct from it, which proves that c_1 and c_2 are distinct inertia lines, and so C_1C_2 is a separation line which must lie in W , since C_1 and C_2 are distinct elements of W .

Now let M be the mean of C_1 and C_2 .

Then as was shown in the remarks at the end of Theorem 171, since C_1C_2 is a separation line lying in W , there is one and only one general plane, say P , passing through M and lying in W to which C_1C_2 is normal.

Also it was proved that this must be an acceleration plane and therefore contains optical and inertia lines.

But an optical or inertia line cannot either lie in a separation plane or be parallel to it and so, by Theorem 153, P and S must have at least one element in common.

Thus, by Theorem 152, P and S must have a general line in common which we shall call a .

Now we have seen that AC_2 is an optical line and C_2 is *after* A , while AC_1 is an inertia line and C_1 is *after* A .

Thus C_1 is not an element of the optical line AC_2 but is *after* an element of it and so there is one single element, say F , which is an element both of the optical line AC_2 and the β sub-set of C_1 .

Thus, since F cannot coincide with C_1 , it must be *before* C_1 , and FC_1 must be an optical line.

But, since M is the mean of C_1 and C_2 , and since C_1C_2 is a separation line, it follows that FM is an inertia line which is normal to C_1C_2 .

Thus, since FM clearly lies in W , it must lie in P .

Now the general line a lies in S and is therefore a separation line and, since it also lies in P , it must intersect FM in some element, say G .

But now since AC_1 is an inertia line while FC_1 is an optical line, it follows that F cannot coincide with A .

Also since AF is an optical line and C_1 lies in the α sub-set of F , it follows that A must lie in the β sub-set of F and so, since A and F are distinct, we must have F after A .

But now since A and G lie in the separation plane S the one is neither before nor after the other.

Now G could not either coincide with F or be after F , for in either case it would be after A , which is impossible.

Thus, since G and F both lie in the inertia line FM , it follows that G is before F .

But, since F is before C_1 , it follows that G is before C_1 .

The element F , however, is the only element common to the inertia line FM and the β sub-set of C_1 and so G does not lie in the β sub-set of C_1 .

It follows that GC_1 is not an optical line and must therefore be an inertia line.

Thus the separation line a and the inertia line GC_1 determine an acceleration plane, say Q .

But now through the element C_1 of the acceleration plane Q there pass two and only two optical lines lying in Q and both of these must intersect a .

Since C_1 does not lie in a , these elements of intersection which we shall call D and E must be distinct, and since O_1C_1 is normal to S it must be normal to O_1D and O_1E .

Thus (O_1, C_1) is an after-conjugate to both (O_1, D) and (O_1, E) and so

$$(O_1, D) \equiv (O_1, X_1),$$

$$(O_1, E) \equiv (O_1, X_1).$$

It follows that D and E are elements of the separation circle whose centre is O_1 .

But now since C_1D is an optical line it follows that C_1D and C_1C_2 must lie either in an optical or an acceleration plane.

Since, however, MD lies in P it must be normal to C_1C_2 and must therefore be either an optical or inertia line.

If, however, MD were an optical line normal to C_1C_2 it could not

intersect any other optical line in the general plane containing MD and C_1C_2 and therefore could not intersect C_1D .

It follows that MD cannot be an optical line and therefore must be an inertia line, and so C_1D and C_1C_2 lie in an acceleration plane.

Thus, since M is the mean of C_1 and C_2 , it follows that C_2D is an optical line.

Similarly C_2E is an optical line.

Thus since O_2C_2 is normal to S and therefore normal to O_2D and O_2E , it follows that (O_2, C_2) is an after-conjugate to both (O_2, D) and (O_2, E) .

Thus we have $(O_2, D) \equiv (O_2, X_2)$

and $(O_2, E) \equiv (O_2, X_2)$,

and so D and E are also elements of the separation circle whose centre is O_2 .

It follows that the two separation circles have the two elements D and E in common.

We can easily see that these are the only elements which they have in common, for if D' be any element common to the two separation circles, then D' must lie in two optical lines passing through C_1 and C_2 and it must also lie in S .

Thus since M is the mean of C_1 and C_2 we must have $D'M$ normal to C_1C_2 and, since $D'M$ lies in W , it must lie in P .

Thus D' must lie in P and, since it also must lie in S , it follows that it must lie in a .

But C_1D and C_1E are the only optical lines which pass through C_1 and intersect a , and so D' must be identical with either D or E .

Thus D and E are the only elements common to the two separation circles.

We have next to prove that D and E lie on opposite sides of AB .

Since A lies inside the separation circle whose centre is O_1 while B lies outside it, it follows that neither A nor B can coincide with either D or E .

Also, by Theorem 194, neither A nor B can lie in DE .

But A and B must both lie in S and, since DE contains all elements common to P and S , it follows that neither A nor B can lie in P .

Let R be the acceleration plane containing AC_2 and BC_2 .

Then since A , B and C_2 all lie in R , but not all in one general line, and since they all lie in W , it follows, by Theorem 149, that R lies in W .

But P and R have the element F in common and therefore, by Theorem 152, they have a general line in common which we shall call f .

Now F is *after* A and *before* C_2 and lies in AC_2 and so F is linearly between A and C_2 .

The general line f passes through F , and since neither A nor B lie in P , while f does lie in P , it follows that f is distinct both from AC_2 and BF .

Thus, since f lies in R , it follows that either f intersects BC_2 in an element linearly between B and C_2 , or else must intersect BA in an element linearly between B and A .

Suppose first, if possible, that f intersects BC_2 in an element K linearly between B and C_2 .

This would mean that BC_2 intersected P in an element K which was *after* B and *before* C_2 .

Then KM would be normal to C_1C_2 , and since it would intersect the optical line BC_2 , it would follow that KM must be an inertia line.

Thus KC_1 would be an optical line, since M is the mean of C_1 and C_2 , and since K is supposed to be *before* C_2 , it would follow that K must be *before* C_1 .

But, since K is supposed to be *after* B , it would follow that C_1 was *after* B , which, as we have already seen, is not so.

Thus f cannot intersect BC_2 in any element linearly between B and C_2 and therefore f must intersect BA in an element H linearly between B and A .

Since H lies in BA and also in P it must lie in DE .

Thus DE intersects BA in an element linearly between B and A .

But now, since C_2 is *after* both A and B , it follows, by Theorem 129 (b), that C_2H is an inertia line and H is *before* C_2 .

Since C_2D and C_2E are optical lines, H cannot coincide with either D or E and so we must have either:

- D linearly between H and E ,
- or E linearly between H and D ,
- or H linearly between D and E .

But since C_2 is *after* D , E and H it would follow, by Theorem 129 (b), in the first case that C_2D must be an inertia line, which we know is not so; while in the second case it would follow that C_2E must be an inertia line which we also know is not so.

Thus we are left with the third alternative, namely that H is linearly between D and E .

It follows that D and E are on opposite sides of AB and so the theorem is proved.

Definition. If O and X_0 be two distinct elements in a separation three-fold W , then the set of all elements in W such as X where

$$(O, X) \equiv (O, X_0)$$

will be called a *separation sphere*.

The element O will be called the *centre* of the separation sphere.

The terms *radius*, *diameter*, *inside*, *outside*, &c. may be defined in a similar manner to the case of a separation circle.

THEOREM 196.

If A_0, A_1 and C be three distinct elements such that A_1 is linearly between A_0 and C and if A_2, A_3, A_4, \dots be elements such that:

A_1 is linearly between A_0 and A_2 ,

A_2 is linearly between A_0 and A_3 ,

.....

.....

and such that: $(A_0, A_1) \equiv (A_1, A_2) \equiv (A_2, A_3) \dots$,

then there are not more than a finite number of the elements A_1, A_2, A_3, \dots linearly between A_0 and C .

It is evident that all the series of elements A_1, A_2, A_3, \dots lie in the general line A_0C which we shall call a .

We shall first prove the theorem for the case where a is an inertia line and C is *after* A_0 .

We shall suppose that a lies in an acceleration plane P .

Now since A_1 is linearly between A_0 and C we must have A_1 *after* A_0 , and so if we take two generators of P of opposite sets passing through A_0 and A_1 respectively, they will intersect in some element, say B_0 , which will be *after* A_0 and *before* A_1 and must lie outside a .

Let b be an inertia line passing through B_0 and parallel to a and let optical lines parallel to A_0B_0 and passing through A_1, A_2, A_3, \dots intersect b in the elements B_1, B_2, B_3, \dots respectively.

Now since A_1 is *after* A_0 and since further:

A_1 is linearly between A_0 and A_2 ,

A_2 is linearly between A_0 and A_3 ,

.....

.....

it follows that:

A_1 is *after* A_0 ; A_2 is *after* A_1 ; A_3 is *after* A_2 ;

Thus since $(A_0, A_1) \equiv (A_1, A_2) \equiv (A_2, A_3) \dots$
 it follows that: A_1 is the mean of A_0 and A_2 ,
 A_2 is the mean of A_1 and A_3 .

But now by construction we have

$$(A_0, A_2) \square (B_0, B_2),$$

and so, since A_1 is the mean of A_0 and A_2 , and since A_1B_1 is parallel to A_0B_0 and A_2B_2 , it follows that B_1 is the mean of B_0 and B_2 and so, by Theorem 98, A_2B_1 is parallel to A_1B_0 .

Similarly A_3B_2 is parallel to A_2B_1 and so on.

Thus, since A_1B_0 is an optical line, it follows that A_2B_1, A_3B_2, \dots are all optical lines and so A_1, A_2, A_3, \dots mark steps taken along a with respect to b .

But since a and b do not intersect, it follows by Post. XVII that C may be surpassed in a finite number of steps taken from A_0 .

Thus there cannot be more than a finite number of the elements A_1, A_2, A_3, \dots linearly between A_0 and C .

Similarly if C be *before* A_0 the same result follows by using Theorem 64 in place of Post. XVII, and so the theorem is proved for the case where a is an inertia line.

Consider next the case where a is a separation line and let b be any inertia line which passes through A_0 .

Then a and b determine an acceleration plane which we shall call P .

Now one of the generators of P which pass through A_1 intersects b in an element which lies in the α sub-set of A_1 , while the other generator intersects b in an element which lies in the β sub-set of A_1 .

Let the former of these generators be called f_1 , and let it intersect b in the element A_1' .

Then since A_1 does not lie in b , it follows that A_1' is *after* A_1 and so, since A_0A_1 is a separation line, we must also have A_1' *after* A_0 .

Let $f_0, f_2, f_3, f_4, \dots$ and f_c be generators of P parallel to f_1 and passing through $A_0, A_2, A_3, A_4, \dots$ and C respectively.

Further let f_2, f_3, f_4, \dots and f_c intersect b in A_2', A_3', A_4', \dots and C' respectively.

Then, since A_1' is *after* A_0 , it follows that f_1 is an after-parallel of f_0 , and since A_1 is linearly between A_0 and C , it follows that f_c is an after-parallel of f_1 and so C' is *after* A_1' .

Further: A_1 is linearly between A_0 and A_2 ,
 A_2 is linearly between A_0 and A_3 ,

Thus we have f_1 is an after-parallel of f_0
 f_2 is an after-parallel of f_1 ,
 f_3 is an after-parallel of f_2 ,

Thus we have A_1 is linearly between A_0 and A_2 ,
 A_2 is linearly between A_1 and A_3 ,
 A_3 is linearly between A_2 and A_4 ,

But since $(A_0, A_1) \equiv (A_1, A_2) \equiv (A_2, A_3) \dots$
 it follows that: A_1 is the mean of A_0 and A_2 ,
 A_2 is the mean of A_1 and A_3 ,

Thus, by Theorem 82,

A_1' is the mean of A_0 and A_2' ,
 A_2' is the mean of A_1' and A_3' ,

and so $(A_0, A_1') \equiv (A_1', A_2') \equiv (A_2', A_3') \dots$

Thus by the first case of the theorem, there cannot be more than a finite number of the elements A_1', A_2', A_3' linearly between A_0 and C' .

But each of these elements which is linearly between A_0 and C' corresponds to one of the series A_1, A_2, A_3, \dots which is linearly between A_0 and C , while any one which is not linearly between A_0 and C' corresponds to one of the series A_1, A_2, A_3, \dots which is not linearly between A_0 and C .

Thus there are not more than a finite number of the elements A_1, A_2, A_3, \dots linearly between A_0 and C , and so the theorem holds when a is a separation line.

As regards the case where a is an optical line and C is *after* A_0 we may proceed just as we have done for the case where a is a separation line.

In this case a is one of the generators of the acceleration plane P , while $f_0, f_1, f_2, \dots, f_c$ will be generators of the opposite set.

The result then follows in a similar manner.

In the case where a is an optical line and C is *before* A_0 we also make use of a similar method except that the element C' in the inertia line b will be *before* A_0 instead of *after* it.

Thus the theorem holds in all cases.

REMARKS.

It will be observed that the above theorem is equivalent to the *Axiom of Archimedes* and has been deduced by the help of Post. XVII.

In our remarks on the introduction of this postulate, its analogy to the Axiom of Archimedes was pointed out together with the fact that the postulate contains no reference to *congruence*.

Having defined congruence of pairs we are able to deduce the Axiom of Archimedes in the usual form as given above.

We shall now give the final postulate of our system which is equivalent to the Axiom of Dedekind.

POSTULATE XXI. **If all the elements of an optical line be divided into two sets such that every element of the first set is before every element of the second set, then there is one single element of the optical line which is not before any element of the first set and is not after any element of the second set.**

Since an element is neither *before* nor *after* itself, it is evident that this one single element may belong either to the first or second set.

Again, if a be an optical line in an acceleration plane P , then through each element of a there passes one single generator of P of the opposite system to that to which a belongs.

Also every such generator intersects a .

Thus there is a one-to-one correspondence between the elements of a and the generators of P of the other system and so it follows that: *if either system of generators of an acceleration plane be divided into two sets such that every generator of the first set is a before-parallel of every generator of the second set, then there is one single generator of the system which is not a before-parallel of any generator of the first set and is not an after-parallel of any generator of the second set.*

Again if b be any inertia or separation line and if P be an acceleration plane containing it, then if we select either system of generators of P , there is a one-to-one correspondence between the elements of b and the generators of the selected system which pass through these elements.

If b be an inertia line and X and Y be any two elements of b , then X will be *before* or *after* Y according as the generator through X is a before- or after-parallel of that through Y .

Thus the property formulated in Post. XXI holds for an inertia line as well as for an optical line.

It is also clear that a corresponding result holds in the case of a separation line, but since here no element is either *before* or *after* another, the property must be formulated somewhat differently.

In order to state the result when b is a separation line we may make a perfectly arbitrary convention with regard to the use of the words *right* and *left*.

Thus if X and Y be any two elements of b , we may say that X is to the *left* or *right* of Y according as the generator of the selected system which passes through X is a before- or after-parallel of that through Y .

We may therefore state the property as follows:

If all the elements of a separation line be divided into two sets such that every element of the first set is to the left of every element of the second set, then there is one single element of the separation line which is not to the left of any element of the first set and is not to the right of any element of the second set.

Definitions. If (A, B) and (C, D) be inertia or optical pairs in which B is *after* A and D *after* C , or if (A, B) and (C, D) be separation pairs, then:

(1) If $(A, B) \equiv (C, D)$ we shall say that the segment AB is *equal* to the segment CD .

(2) If $(A, B) \equiv (C, E)$, where E is any element linearly between C and D , we shall say that the segment AB is *less than* the segment CD .

(3) If $(A, B) \equiv (C, F)$, where F is any element such that D is linearly between C and F , we shall say that the segment AB is *greater than* the segment CD .

In the case of separation or inertia segments we must always have either:

AB is equal to CD ,

or AB is less than CD ,

or AB is greater than CD .

In the case of optical segments, however, this is only true provided they lie in the same, or in parallel optical lines.

Again if (A, B) and (C, D) be inertia or optical pairs in which B is

after A and D after C , or if they be separation pairs, and if E, F, G be elements such that F is linearly between E and G while

$$(A, B) \equiv (E, F)$$

and

$$(C, D) \equiv (F, G),$$

we shall say that *the length of the segment EG is equal to the sum of the lengths of the segments AB and CD .*

It is evident that the lengths of two optical segments can only have a sum in this sense provided they lie in the same or in parallel optical lines, whereas the lengths of two inertia segments or two separation segments always have a sum.

Having thus introduced the idea of the length of a segment being equal to the sum of the lengths of two others we can obviously have any *multiple* and also (as follows from the remarks at the end of Theorem 82) any *sub-multiple* of a given segment: using the terms "multiple" and "sub-multiple" in the ordinary sense.

We may also clearly have a segment equal to any proper or improper fractional part of the given segment.

Again, if A_0 and A_1 be two distinct elements in a general line l , it may easily be shown that there are elements $A_2, A_3, A_4 \dots A_n \dots$ in l on the same side of A_0 as is A_1 and such that the segment A_0A_n is equal to n times the segment A_0A_1 .

Similarly there are elements $A_{-1}, A_{-2}, A_{-3} \dots A_{-n} \dots$ lying in l but on the opposite side of A_0 and such that the segment $A_{-n}A_0$ is equal to n times the segment A_0A_1 .

Again, it may easily be shown that corresponding to any positive rational number $r = \frac{p}{q}$ there is an element A_r in l on the same side of A_0 as is A_1 and such that q times the segment A_0A_r is equal to p times the segment A_0A_1 .

Similarly, corresponding to any negative rational number $-r = -\frac{p}{q}$, it may be shown that there is an element A_{-r} in l , but on the opposite side of A_0 and such that q times the segment $A_{-r}A_0$ is equal to p times the segment A_0A_1 .

By making use of our equivalents of the Archimedes and Dedekind axioms for the elements of a general line along with the corresponding properties of *real numbers*, it is possible in this way to set up a one-to-one correspondence between the elements of a general line and the aggregate of *real numbers*.

The logical steps involved in setting up such a correspondence have

been carefully investigated by others and it is unnecessary to go into further details here.

These may be found, for instance, in Pierpont's *Theory of Functions of Real Variables*, vol. I, chapters I and II, and in other works.

The absolute value of the difference of the real numbers corresponding to the two ends of any segment of l gives us a real number which may be called *the numerical value of the length* of the segment in terms of the unit segment A_0A_1 .

If l be an inertia or separation line, the length of any segment of a general line of the same kind as l is always expressible in terms of our selected unit segment; but this is not true in general if l be an optical line.

It is to be observed that the length of an inertia or optical segment is independent of which end of the segment is *after* the other.

In this respect the question of the equality of inertia or optical segments differs from the closely related question of the congruence of the inertia or optical pairs forming the ends of the segments.

The criterion of *proportion* given by Euclid and which, according to Sir T. L. Heath, is probably due to Eudoxos, is clearly applicable in our geometry.

Certain results regarding the proportion of segments may easily be shown to hold for all types of general line.

Thus if O, A, B be the corners of any general triangle, and if C be any element distinct from A in the general half-line OA , then if a general line through C parallel to AB intersect OB in the element D , we must have

$$\text{segment } OA : \text{segment } OC = \text{segment } OB : \text{segment } OD.$$

This may be proved by the method of De Morgan described by Heath in his edition of Euclid, vol. II, p. 124.

If OA should be a separation line and OB an inertia line normal to OA (or conversely), while AB is an optical line, then CD will also be an optical line and so it follows that: *separation segments are proportional to their conjugate inertia segments.*

Again if through D a general line be taken parallel to OA and intersecting AB in the element E , while through E a general line is taken parallel to OB and intersecting OA in the element F , we clearly have

$$(C, D) \equiv (A, E)$$

and

$$(A, C) \equiv (F, O).$$

Thus by Theorem 185

$$(A, F) \equiv (C, O).$$

But since EF is parallel to BO it follows that:

$$\text{segment } AO : \text{segment } AF = \text{segment } AB : \text{segment } AE.$$

Thus since $\text{segment } AF = \text{segment } CO$

and $\text{segment } AE = \text{segment } CD$,

we have

$$\text{segment } AO : \text{segment } CO = \text{segment } AB : \text{segment } CD.$$

THEOREM 197.

The geometry of a separation threefold is formally identical with the ordinary (Euclidean) geometry of three dimensions.

This theorem may be proved by showing that in a separation threefold a set of propositions hold which have already been demonstrated to be sufficient as a basis for the ordinary three dimensional (Euclidean) geometry.

We have already seen that various axioms of Peano, not involving the idea of congruence, hold generally in our system, and not merely in a separation threefold. A separation threefold, however, being the only form of general threefold in which all general lines are of one type and all general planes also of one type, is evidently the only one in which we could hope to find formal identity with Euclidean geometry.

It is proposed here to give a set of propositions which will be found to be equivalent to a set of assumptions given by Veblen, and shown by him to be sufficient for the deduction of ordinary geometry.

Veblen speaks of three points being in the "order" $\{ABC\}$, meaning thereby that the three points are related in a manner analogous to that of three elements A, B, C in our geometry where B is "*linearly between*" A and C .

The existence, however, of optical and inertia lines in which two elements are asymmetrically related makes it undesirable to speak of elements in a separation line being in the "order" $\{ABC\}$; since this seems to suggest that B is *after* A and C *after* B .

The expression "*linearly between*" carries no such suggestion and, as it serves the purpose equally well, it will be employed here.

Let us consider the set of elements lying in a separation threefold W .

We know that every general line in W is a separation line and every general plane in W is a separation plane.

If now we consider the definition of an element B being "*linearly between*" the elements A and C , we see that:

(1) If the element B be linearly between the elements A and C , then A , B and C must be distinct and also B is linearly between the elements C and A .

From this same definition in conjunction with the remark at the end of Theorem 42, it follows that:

(2) If the element B be linearly between the elements A and C , then C is not linearly between the elements B and A .

From the definition of a separation line, it follows at once by Post. XIII that:

(3) If the two distinct elements C and D lie in the separation line AB , then A lies in the separation line CD .

From Theorem 43 and the definition of "*linearly between*," it follows that:

(4) If A and B be two distinct elements in W , there exists in W at least one element C such that B is linearly between A and C .

If three distinct elements A , B , C do not all lie in one general line, and if D be an element such that C is linearly between B and D , it is evident that A , B and D cannot lie in one general line.

From this, in conjunction with Theorem 115, it follows directly that:

(5) If three distinct elements A , B and C lie in W , but not in the same separation line, and if D and E are two elements such that C is linearly between B and D and E is linearly between C and A , then there exists in W an element F , such that F is linearly between A and B , and such that D , E and F lie in the same separation line.

Since all the elements of W do not lie in one separation line, it follows that:

(6) There exist three distinct elements in W , say A , B , C , such that B is not linearly between A and C , and C is not linearly between B and A , and A is not linearly between C and B .

If A and B be two distinct elements in W , and if a be any separation line in W while C is any element in a , then we have seen that there is one single element, say D' , such that:

$$(A, B) \equiv (C, D');$$

also we have seen that there are two and only two elements in a , say D_1 and D_2 , such that:

$$(C, D_1) \equiv (C, D')$$

and

$$(C, D_2) \equiv (C, D').$$

and accordingly

$$(A, B) \equiv (C, D_1)$$

and

$$(A, B) \equiv (C, D_2).$$

But these elements D_1 and D_2 lie on opposite sides of C , and therefore there is only one of these elements in either of the separation half-lines into which C divides a . Thus:

(7) If A and B be any two distinct elements in W and if C be the end of any separation half-line in W , there exists one and only one element D in this separation half-line, such that:

$$(A, B) \equiv (C, D).$$

Since we have shown that the relation of congruence for separation pairs is a transitive relation, it follows that:

(8) If (A, B) , (C, D) , (E, F) be separation pairs in W , such that:

$$(A, B) \equiv (C, D)$$

and

$$(C, D) \equiv (E, F),$$

then we have

$$(A, B) \equiv (E, F). \bullet$$

In Theorem 184 we proved that:

(9) If (A_1, B_1) , (A_2, B_2) , (B_1, C_1) , (B_2, C_2) be separation pairs, such that:

$$(A_1, B_1) \equiv (A_2, B_2)$$

and

$$(B_1, C_1) \equiv (B_2, C_2),$$

then if B_1 be linearly between A_1 and C_1 , and if B_2 be linearly between A_2 and C_2 , we shall also have

$$(A_1, C_1) \equiv (A_2, C_2).$$

On page 294 we proved that:

(10) If (A, B) be any separation pair, then

$$(A, B) \equiv (B, A).$$

In Theorem 193 we proved that:

(11) If A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1 , and A_2, B_2, C_2 be the corners of a triangle in a separation plane P_2 , and if D_1 be an element in B_1C_1 such that C_1 is linearly between B_1 and D_1 , while D_2 is an element in B_2C_2 such that C_2 is linearly between B_2 and D_2 ; and if, further,

$$(A_1, B_1) \equiv (A_2, B_2),$$

$$(B_1, C_1) \equiv (B_2, C_2),$$

$$(C_1, A_1) \equiv (C_2, A_2),$$

$$(B_1, D_1) \equiv (B_2, D_2),$$

then we shall also have

$$(A_1, D_1) \equiv (A_2, D_2).$$

In Theorem 195 we proved that:

(12) If a separation circle in a separation plane S pass through an element A which is inside and another element B which is outside a second separation circle in S , then the two separation circles have two elements in common which lie on opposite sides of the separation line AB .

In Theorem 113 we proved that:

(13) If e be a general line in a separation plane, and if A be any element of the separation plane which does not lie in e , then there is one single general line through A in the separation plane which does not intersect e .

In Theorem 196 we proved that:

(14) If A_0, A_1 and C be three distinct elements such that A_1 is linearly between A_0 and C , and if A_2, A_3, A_4, \dots be elements such that:

A_1 is linearly between A_0 and A_2 ,

A_2 is linearly between A_0 and A_3 ,

.....

.....

and such that:

$$(A_0, A_1) \equiv (A_1, A_2) \equiv (A_2, A_3) \dots,$$

then there are not more than a finite number of the elements A_1, A_2, A_3, \dots linearly between A_0 and C .

Since all the elements of a separation threefold do not lie in one separation plane, it follows that:

(15) If A, B, C be three distinct elements in W which do not all lie in one separation line, there exists an element D in W which does not lie in the same separation plane with A, B and C .

From Theorem 152 it follows that:

(16) Two distinct separation planes in W , which have one element in common, have two elements in common.

Veblen has shown that a set of assumptions formally similar to the above sixteen propositions are a sufficient basis for elementary geometry as given by Euclid.

Euclid however only considers constructions which involve the straight line and circle, and in order to treat space as a continuum, it is necessary to add an assumption equivalent to the axiom of Dedekind.

The Dedekind property is secured in our geometry by Post. XXI as already pointed out.

In showing that ordinary geometry may be deduced from a set of assumptions formally equivalent to the above, Veblen makes use of definitions of a straight line and plane which are different from, but clearly consistent with, ours.

His definition of a straight line, as modified to apply to a separation line in W , is equivalent to the following:

If A and B be two distinct elements in W , then the separation line AB consists of A and B together with all elements such as X , where either:

- B is linearly between A and X ,
- or X is linearly between A and B ,
- or A is linearly between X and B .

This clearly follows from Theorem 44, in conjunction with the definition of "linearly between."

Veblen's definition of a plane as modified to apply to a separation plane in W is equivalent to the following:

If A , B and C be three distinct elements in W which do not all lie in one separation line, then the separation plane containing A , B and C consists of all elements which lie in separation lines which contain pairs of elements in the linear intervals AB , BC , CA .

From Theorem 112 it follows that all such elements as those mentioned lie in the separation plane containing A , B and C , while by Theorem 128 (1) it is easy to see that any element in the separation plane containing A , B and C must lie in a separation line containing a pair of elements in the linear intervals AB , BC , CA .

For let P be the separation plane containing A , B and C , and let X be any element in it, while D is any element distinct from X and linearly between, say, B and C .

Then, by Theorem 128 (1), XD must either be identical with BC or with AD , or must intersect AC in an element linearly between A and C , or else must intersect AB in an element linearly between A and B .

In all these cases XD is a separation line containing a pair of elements in the linear intervals AB , BC , CA .

Thus Veblen's definition of a plane is consistent with our definition of a separation plane.

Further, Veblen's other definitions are also clearly consistent with ours in so far as a separation threefold is concerned, and so the geometry of a separation threefold is formally identical with the ordinary (Euclidean) geometry of three dimensions; as was to be proved.

REMARKS.

From this stage on, we are evidently at liberty to make use of any known theorem of ordinary geometry, and apply it in the geometry of a separation threefold or separation plane, since any separation plane may always be taken as lying in a separation threefold.

For details as to how this is to be done, the reader is referred to the works of Veblen and others*.

It is there shown how angles, &c. may be defined, together with various other points which we here omit.

It will be seen that the definition there given of intersecting straight lines being at right angles to one another is in agreement with our definition of intersecting separation lines being "normal" to one another.

Accordingly if A, B, C be the corners of a triangle in a separation plane, such that BC is normal to AC , then the Theorem of Pythagoras shows that:

$$(AB)^2 = (BC)^2 + (AC)^2.$$

THEOREM 198.

If B and C be two distinct elements in a separation line and O be their mean, and if A be any element in an optical line a which passes through O and is normal to BC , then

$$(A, B) \equiv (A, C).$$

Since a is an optical line which is normal to the separation line BC , it follows that a and BC lie in an optical plane, say P .

If the element A should happen to coincide with O then, since BC is a separation line, the theorem obviously holds.

Suppose next that A does not coincide with O , and let d be a separation line passing through O and normal to P .

Then a and d determine an optical plane, say Q , which is completely normal to P ; and, since BC and d are both separation lines and are normal to one another, it follows that they lie in a separation plane, say S .

Let D be any element of d distinct from O .

Then DO is normal to BC and so, by Theorem 186, we have

$$(D, B) \equiv (D, C).$$

* See, for instance, the article by Veblen on the Foundations of Geometry in *Monographs on Modern Mathematics*, edited by J. W. A. Young, Longmans, 1911. Also references given in this article.

But since Q is completely normal to P , it follows that DA is normal to P and so DA is normal to both AB and AC .

Also, since D is not an element of a , and a is a generator of the optical plane Q , it follows that DA must be a separation line.

Similarly, since B and C are not elements of a , and a is a generator of the optical plane P , it follows that both BA and CA are separation lines.

Thus DA and BA must lie in a separation plane, say R_1 , and DA and CA must lie in a separation plane, say R_2 .

Thus, by Theorem 190, since

$$(A, D) \equiv (A, D)$$

and

$$(D, B) \equiv (D, C),$$

it follows that:

$$(A, B) \equiv (A, C)$$

as was to be proved.

THEOREM 199.

If O and X_0 be two distinct elements in a separation line lying in an optical plane P , then the set of all elements in P such as X where OX is a separation line and

$$(O, X) \equiv (O, X_0)$$

consists of a pair of parallel optical lines.

Let X_0' be an element in OX_0 and on the opposite side of O to that on which X_0 lies, and such that:

$$(O, X_0') \equiv (O, X_0).$$

Then X_0' is an element of the set we are considering, and it is evident that it is the only one besides X_0 lying in the separation line OX_0 .

Further it is evident that O is the mean of X_0 and X_0' .

Let a , b and c be three generators of the optical plane P passing through X_0 , X_0' and O respectively.

Let X_1 be any element in a distinct from X_0 , and let OX_1 intersect b in X_1' .

Further, let c intersect $X_0'X_1$ in the element M .

Then OX_1 and $X_0'X_1$ are both separation lines, since they have each got elements in two distinct generators of the optical plane.

Now since c must be parallel to a , and since O is the mean of X_0 and X_0' , it follows by Theorem 92 that M is the mean of X_1 and X_0' .

But, since OM is an optical line and $X_0'X_1$ is a separation line in the same optical plane P with it, it follows that OM is normal to $X_0'X_1$,

Thus by Theorem 198 we must have

$$(O, X_1) \equiv (O, X_0').$$

But, since

$$(O, X_0') \equiv (O, X_0),$$

it follows that:

$$(O, X_1) \equiv (O, X_0).$$

Similarly

$$(O, X_1') \equiv (O, X_0'),$$

and so

$$(O, X_1') \equiv (O, X_0).$$

Thus X_1 and X_1' are evidently elements of the set we are considering, and are clearly the only ones lying in the separation line OX_1 .

Similarly any other separation line passing through O and lying in P will intersect a and b in elements belonging to the set considered, and these will be the only ones lying in that separation line.

Thus the parallel optical lines a and b together constitute the set of elements in P , such as X , where OX is a separation line and

$$(O, X) \equiv (O, X_0),$$

and so the theorem is proved.

REMARKS.

Certain interesting results follow directly from the last theorem.

Thus if we consider any triangle in an optical plane whose corners are A , B and C , then not more than one of the general lines AB , BC , CA can be an optical line, since no two optical lines in an optical plane can intersect.

If BC be an optical line, then AB and CA must be separation lines, and from the last theorem it follows that:

$$(A, B) \equiv (A, C).$$

If, on the other hand, neither AB , BC nor CA be an optical line, they must all be separation lines.

In this case, let a , b and c be generators of the optical plane passing through A , B and C respectively, and intersecting BC , CA and AB in A' , B' and C' respectively.

Then since neither AB , BC nor CA are optical lines, it follows that neither A' , B' nor C' can coincide with a corner of the triangle.

Thus we must either have

$$(1) \quad A' \text{ linearly between } B \text{ and } C,$$

$$\text{or} \quad (2) \quad C \text{ linearly between } A' \text{ and } B,$$

$$\text{or} \quad (3) \quad B \text{ linearly between } C \text{ and } A'.$$

In the first case, we shall also have

A linearly between B' and C ,

and

A linearly between B and C' .

In the second case, we shall also have

C linearly between A and B' ,

and

C' linearly between A and B .

In the third case, we shall also have

B linearly between C' and A ,

and

B' linearly between C and A .

Thus in all cases one of the three elements A' , B' , C' , and only one, lies linearly between a pair of the corners A , B , C .

Now let us consider the case, for instance, where A' is linearly between B and C .

It follows directly from the last theorem that :

$$(B, A) \equiv (B, A'),$$

and

$$(C, A) \equiv (C, A').$$

This remarkable result may be expressed as follows :

If all three sides of a triangle in an optical plane be separation segments, then the sum of the lengths of a certain two of the sides is equal to that of the third side.

Again, if a and b be a pair of neutral-parallel optical lines and if A_1 and A_2 be any elements in a , while B_1 and B_2 are any elements in b , we have

$$(A_1, B_1) \equiv (A_1, B_2),$$

and

$$(B_2, A_1) \equiv (B_2, A_2).$$

Thus we see that we must have

$$(A_1, B_1) \equiv (A_2, B_2).$$

It will be observed that, in the case of an optical plane, a pair of parallel optical lines is the analogue of a circle, in so far as any analogue exists.

Again, if W be an optical threefold and O be any element in it, while c is the generator of W which passes through O , then any general plane in W which contains c is an optical plane, while any one which passes through O , but does not contain c , is a separation plane.

If then S be any separation plane lying in W and passing through O and X_0 be any element in it distinct from O , the set of elements in S , such as X , where

$$(O, X) \equiv (O, X_0),$$

constitutes a separation circle.

If through each element of the separation circle a generator of W be taken, then any element X on any such generator will also satisfy the relation

$$(O, X) \equiv (O, X_0).$$

Further, it is clear that no other element of W does satisfy it.

The set of elements thus obtained lie on a sort of cylinder which, in the case of an optical threefold, takes the place of a sphere.

We shall call this an *optical circular cylinder*.

THEOREM 200.

If A_1, B_1, C_1 be the corners of a triangle in an acceleration plane P_1 and A_2, B_2, C_2 be the corners of a triangle in an acceleration plane P_2 , and if further B_1C_1 be a separation line which is normal to the inertia line A_1C_1 , while B_2C_2 is a separation line which is normal to the inertia line A_2C_2 , then :

$$(1) \text{ If } (C_1, A_1) \equiv (C_2, A_2)$$

$$\text{and } (C_1, B_1) \equiv (C_2, B_2),$$

$$\text{we shall either have } (A_1, B_1) \equiv (A_2, B_2),$$

or else both A_1B_1 and A_2B_2 will be optical lines.

$$(2) \text{ If } (A_1, C_1) \equiv (C_2, A_2)$$

$$\text{and } (C_1, B_1) \equiv (C_2, B_2),$$

$$\text{we shall either have } (A_1, B_1) \equiv (B_2, A_2),$$

or else both A_1B_1 and B_2A_2 will be optical lines.

Consider first part (1) of the theorem.

Since $(C_1, A_1) \equiv (C_2, A_2)$, and since these are inertia pairs, we must have either A_1 before C_1 and A_2 before C_2 , or else have A_1 after C_1 and A_2 after C_2 .

We shall only consider the case where A_1 is before C_1 and A_2 before C_2 , since the other case is quite analogous.

If A_1B_1 were an optical line we should have (C_1, A_1) a before-conjugate to (C_1, B_1) , and if A_2' were an element in C_2A_2 , such that

(C_2, A_2') were a before-conjugate to (C_2, B_2) , then it would follow by Theorem 181 that we must have

$$(C_1, A_1) \equiv (C_2, A_2'),$$

and so we should have

$$(C_2, A_2) \equiv (C_2, A_2').$$

Thus A_2' would be identical with A_2 , and so $A_2 B_2$ would also be an optical line.

We are not, however, at liberty to assert in this case that

$$(A_1, B_1) \equiv (A_2, B_2),$$

but only that they are both optical pairs.

We shall suppose next that $A_1 B_1$ is not an optical line, and that accordingly $A_2 B_2$ is not an optical line.

Let D_1 and D_2 be elements in $C_1 A_1$ and $C_2 A_2$ respectively, such that (C_1, D_1) is a before-conjugate to (C_1, B_1) , and (C_2, D_2) is a before-conjugate to (C_2, B_2) .

Then, by Theorem 181, we must have

$$(C_1, D_1) \equiv (C_2, D_2).$$

Now two cases occur; we may have

- (1) A_1 linearly between D_1 and C_1 ,
or (2) D_1 linearly between A_1 and C_1 .

In the first case, since we also have

$$(C_1, A_1) \equiv (C_2, A_2),$$

it follows that we must also have A_2 linearly between D_2 and C_2 , as was shown in the remarks at the end of Theorem 188.

Similarly, in the second case we must also have D_2 linearly between A_2 and C_2 .

Again, in the first case we have D_1 *before* C_1 , and must therefore have A_1 *after* D_1 and *before* C_1 .

But A_1 could not be *before* B_1 , for then A_1 would require to lie in the optical line $D_1 B_1$, which we know is not the case.

Further A_1 could not be *after* B_1 , for then, since C_1 is *after* A_1 , we should have C_1 *after* B_1 contrary to the hypothesis that $B_1 C_1$ is a separation line.

It follows that in case (1) $A_1 B_1$ is a separation line, and similarly $A_2 B_2$ is a separation line.

In case (2), on the other hand, we must have A_1 *before* D_1 and so, since D_1 is *before* B_1 , we must have A_1 *before* B_1 .

Thus, since $D_1 A_1$ is an inertia line, and since D_1 is the only element common to it and the β sub-set of B_1 , it follows in this case that $A_1 B_1$ is an inertia line, and similarly $A_2 B_2$ is an inertia line.

We shall consider cases (1) and (2) separately.

Case (1).

We have here got $A_1 B_1$ and $A_2 B_2$, both separation lines.

Now let W_1 and W_2 be rotation threefolds containing P_1 and P_2 respectively, and let S_1 and S_2 be the separation planes in W_1 and W_2 which pass through C_1 and C_2 , and are normal to the inertia lines $A_1 C_1$ and $A_2 C_2$ respectively.

Then since $B_1 C_1$ is normal to $A_1 C_1$, it follows that $B_1 C_1$ must lie in S_1 and similarly $B_2 C_2$ must lie in S_2 .

Now since $A_1 B_1$ is a separation line, there is an acceleration plane which passes through A_1 , lies in W_1 and is normal to $A_1 B_1$.

This acceleration plane contains two optical lines which pass through A_1 and must be normal to $A_1 B_1$ and which must intersect S_1 , since S_1 is a separation plane in the same rotation threefold along with these optical lines.

Let one of these optical lines intersect S_1 in the element E_1 .

Similarly we can show that there are two optical lines passing through A_2 and lying in W_2 , and which are normal to $A_2 B_2$.

These optical lines may be shown in a similar manner to intersect S_2 , and we shall suppose that one of them intersects S_2 in the element E_2 .

Now since the optical line $A_1 E_1$ is normal to the separation line $A_1 B_1$, it follows that $A_1 E_1$ and $A_1 B_1$ lie in an optical plane.

Similarly $A_2 E_2$ and $A_2 B_2$ lie in an optical plane.

But, since an optical line in an optical plane is normal to every general line in the optical plane, it follows that $A_1 E_1$ is normal to $E_1 B_1$, and similarly $A_2 E_2$ is normal to $E_2 B_2$.

Again, since $E_1 B_1$ lies in S_1 and since S_1 is normal to $A_1 C_1$, it follows that $A_1 C_1$ is normal to $E_1 B_1$.

Similarly $A_2 C_2$ is normal to $E_2 B_2$.

Thus $E_1 B_1$ is normal to the two intersecting general lines $A_1 E_1$ and $A_1 C_1$, and is therefore normal to the general plane containing them.

Similarly $E_2 B_2$ is normal to the general plane containing $A_2 E_2$ and $A_2 C_2$.

It follows that $E_1 B_1$ is normal to $E_1 C_1$, and that $E_2 B_2$ is normal to $E_2 C_2$.

Again since S_1 is normal to $A_1 C_1$ it follows that $E_1 C_1$ is normal to $A_1 C_1$, and similarly it follows that $E_2 C_2$ is normal to $A_2 C_2$.

Thus since A_1C_1 and A_2C_2 are inertia lines while A_1E_1 and A_2E_2 are optical lines, it follows that (C_1, E_1) and (C_2, E_2) are after-conjugates to (C_1, A_1) and (C_2, A_2) respectively.

But since $(C_1, A_1) \equiv (C_2, A_2)$,

it follows by Theorem 181 that:

$$(C_1, E_1) \equiv (C_2, E_2).$$

Thus C_1, B_1, E_1 are the corners of a triangle in the separation plane S_1 and C_2, B_2, E_2 are the corners of a triangle in the separation plane S_2 , while further

$$(E_1, C_1) \equiv (E_2, C_2),$$

$$(C_1, B_1) \equiv (C_2, B_2),$$

and also B_1E_1 is normal to C_1E_1 and B_2E_2 is normal to C_2E_2 , and so, by Theorem 190,

$$(E_1, B_1) \equiv (E_2, B_2).$$

But since E_1B_1 and A_1B_1 are separation lines lying in an optical plane, of which A_1E_1 is a generator, it follows from the remarks at the end of Theorem 199 that:

$$(E_1, B_1) \equiv (A_1, B_1).$$

Similarly $(E_2, B_2) \equiv (A_2, B_2)$.

Thus we get finally $(A_1, B_1) \equiv (A_2, B_2)$,

and so the theorem is proved in case (1).

Case (2).

We have here got A_1B_1 and A_2B_2 , both inertia lines.

As before, let W_1 and W_2 be rotation threefolds containing P_1 and P_2 respectively, and let S_1 and S_2 be the separation planes in W_1 and W_2 which pass through C_1 and C_2 and are normal to the inertia lines A_1C_1 and A_2C_2 respectively.

Then, as in the first case, B_1C_1 lies in S_1 and B_2C_2 lies in S_2 .

Let b_1 be the separation line in S_1 which passes through B_1 and is normal to B_1C_1 , and similarly let b_2 be the separation line in S_2 which passes through B_2 and is normal to B_2C_2 .

Then, since A_1B_1 is an inertia line, it follows that A_1B_1 and b_1 lie in an acceleration plane, say Q_1 , and similarly A_2B_2 and b_2 lie in an acceleration plane, say Q_2 .

Let one of the generators of Q_1 which pass through A_1 intersect b_1 in the element F_1 , and let one of the generators of Q_2 which pass through A_2 intersect b_2 in the element F_2 .

Now since A_1C_1 is an inertia line, it follows that A_1C_1 and A_1F_1 determine an acceleration plane, and similarly A_2C_2 and A_2F_2 determine an acceleration plane.

Since C_1F_1 lies in S_1 it must be normal to A_1C_1 , and since C_2F_2 lies in S_2 it must be normal to A_2C_2 .

Thus, since A_1F_1 and A_2F_2 are optical lines, it follows that (C_1, F_1) , (C_2, F_2) are after-conjugates to (C_1, A_1) and (C_2, A_2) respectively, and so since

$$(C_1, A_1) \equiv (C_2, A_2),$$

it follows by Theorem 181 that :

$$(C_1, F_1) \equiv (C_2, F_2).$$

But now C_1, F_1, B_1 are the corners of a triangle in the separation plane S_1 and C_2, F_2, B_2 are the corners of a triangle in the separation plane S_2 , while further

$$(C_1, B_1) \equiv (C_2, B_2),$$

$$(C_1, F_1) \equiv (C_2, F_2),$$

and also F_1B_1 is normal to C_1B_1 and F_2B_2 is normal to C_2B_2 and so, by Theorem 190,

$$(B_1, F_1) \equiv (B_2, F_2).$$

Since F_1B_1 lies in S_1 it is normal to A_1C_1 , and by hypothesis it is also normal to B_1C_1 and so, since A_1C_1 and B_1C_1 are intersecting general lines in P_1 , it follows that F_1B_1 is normal to P_1 .

Thus F_1B_1 must be normal to A_1B_1 and similarly F_2B_2 must be normal to A_2B_2 .

But A_1F_1 and A_2F_2 are optical lines while A_1B_1 and A_2B_2 are inertia lines and so (B_1, F_1) and (B_2, F_2) are after-conjugates to (B_1, A_1) and (B_2, A_2) respectively.

Thus since $(B_1, F_1) \equiv (B_2, F_2)$,

it follows, by Theorem 181, that :

$$(B_1, A_1) \equiv (B_2, A_2),$$

that is

$$(A_1, B_1) \equiv (A_2, B_2),$$

as was to be proved.

Consider now part (2) of the theorem.

Since $(A_1, C_1) \equiv (C_2, A_2)$ and since these are inertia pairs we must either have A_1 before C_1 and A_2 after C_2 or else have A_1 after C_1 and A_2 before C_2 .

There is then no difficulty in showing that :

$$(A_1, B_1) \equiv (B_2, A_2),$$

provided that A_1B_1 be not an optical line.

The proof is quite analogous to that of the first part of the theorem except that we make use of the result given at the end of Theorem 181 in place of Theorem 181 itself.

It is also evident that if A_1B_1 be an optical line, then B_2A_2 must also be an optical line.

Thus both parts of the theorem hold.

It will be observed that the two parts of Theorem 200 are the analogue for acceleration planes of Theorem 189.

THEOREM 201.

(1) *If B and C be two distinct elements in a separation line and O be their mean and if A be any element in an inertia line a which passes through O and is normal to BC, then either*

$$(A, B) \equiv (A, C),$$

or else both AB and AC are optical lines.

(2) *If B and C be two distinct elements in an inertia line and O be their mean and if A be any element in a separation line a which passes through O and is normal to BC, then either*

$$(B, A) \equiv (A, C),$$

or else both BA and AC are optical lines.

The two parts of this theorem follow directly from the last and no further proofs are required.

It will be observed that they form the analogue of Theorem 186.

THEOREM 202.

If A, B, C be three distinct elements in an acceleration plane P which do not all lie in one general line and if

$$(A, B) \equiv (A, C),$$

or if

$$(B, A) \equiv (A, C),$$

then BC cannot be an optical line.

Since the only congruence of optical pairs is co-directional, it is evident that neither AB nor AC can be optical lines and must therefore be either inertia or separation lines.

Consider first the case where they are inertia lines and

$$(A, B) \equiv (A, C).$$

It is evident that we must either have A *before* both B and C or *after* both B and C .

Suppose A is *before* both B and C and let a be a separation line passing through A and normal to the acceleration plane containing AB and AC .

Let D be an element in a such that (A, D) is a before-conjugate to (A, B) .

Then (A, D) will also be a before-conjugate to (A, C) since

$$(A, B) \equiv (A, C).$$

Thus DB and DC will both be optical lines, and so BC cannot be an optical line*.

If A be *after* both B and C the result follows in a similar manner.

Next consider the case where AB and AC are inertia lines but where

$$(B, A) \equiv (A, C).$$

We must then either have A *after* B and C *after* A or else A *before* B and C *before* A .

In either case it is evident that BC could not be an optical line, for otherwise A would be *after* one element of it and *before* another and yet not lie in the optical line; which we know to be impossible.

Consider next the case where AB and AC are separation lines and where accordingly

$$(A, B) \equiv (A, C)$$

implies

$$(B, A) \equiv (A, C),$$

and conversely.

Now we know that there is one single optical parallelogram in P having A as centre and B as one of its corners.

Suppose, if possible, that BC is an optical line which we shall denote shortly by b .

Then b would be one of the side lines of this optical parallelogram, and we shall denote the opposite side line by b' .

Let B' be the corner opposite to B and let D and D' be the remaining two corners: D lying in b and D' lying in b' .

Let CA intersect b' in the element C' and let optical lines passing through C and C' respectively and parallel to BD' intersect b' and b in E' and E respectively.

Then E', C, E, C' would form the corners of an optical parallelogram having also b and b' as a pair of opposite side lines.

* It is also to be noted that B is neither *before* nor *after* C in this case.

Thus, since the diagonal line CC' passes through A , it follows, by Theorem 62, that these two optical parallelograms would have a common centre A .

But now either (A, D) or (A, D') would be an after-conjugate to (A, B) while (A, E) or (A, E') would be an after-conjugate to (A, C) and DE and $D'E'$ would both be optical lines.

Thus by the first case of the theorem it is impossible that we should have

$$(A, D) \equiv (A, E)$$

or

$$(A, D') \equiv (A, E').$$

If however we had $(A, B) \equiv (A, C)$,

these other congruences would require to hold and so it is impossible to have BC an optical line if

$$(A, B) \equiv (A, C).$$

Thus the theorem holds in all cases.

It is important to note that while this result holds for an acceleration plane, it does not, as we have already shown, hold for an optical plane.

Thus since an optical line can only lie in an acceleration or optical plane, it follows that:

If B and C be two distinct elements in an optical line while A is an element which does not lie in BC , then if

$$(A, B) \equiv (A, C)$$

the elements A, B, C must lie in an optical plane.

THEOREM 203.

If A, B and C be three distinct elements which lie in an acceleration plane P , but do not all lie in one general line, then:

(1) *If BC be a separation line and O be the mean of B and C and if*

$$(A, B) \equiv (A, C),$$

or if AB and AC be both optical lines, we must have AO normal to BC .

(2) *If BC be an inertia line and O be the mean of B and C and if*

$$(B, A) \equiv (A, C),$$

or if BA and AC be both optical lines, we must have AO normal to BC .

Let us consider the first part of the theorem.

If AB and AC are both optical lines, then A , B and C are three corners of an optical parallelogram of which O is the centre and so AO is normal to BC by definition.

We shall therefore suppose that:

$$(A, B) \equiv (A, C),$$

and accordingly that AB and AC are either both inertia lines or both separation lines.

Let d be a separation line passing through A and normal to P .

Then P and d determine a rotation threefold W .

Let e be a separation line in W which passes through O and is normal to BC , but which is not parallel to d .

Then BC and e , being separation lines which intersect and are normal to one another, must lie in a separation plane, say S , which must lie in W .

Now since S does not contain the separation line which passes through O and is parallel to d , and since S and d both lie in W , but d does not lie in S , it follows that d must intersect S in some element, say D .

Now AD being normal to P must be normal to both AB and AC and, since these are either both inertia lines or both separation lines, it follows that AD and AB on the one hand and AD and AC on the other must lie in acceleration planes or else in separation planes.

Also, since S is a separation plane, it follows that DB and DC are both separation lines.

Thus since $(A, B) \equiv (A, C)$

and $(A, D) \equiv (A, D)$,

it follows by Theorem 189 or by Theorem 200 (according as AB and AC are separation lines or inertia lines) that:

$$(B, D) \equiv (C, D).$$

But now since D , B , and C are three distinct elements in the separation plane S which do not all lie in one general line, and since O is the mean of B and C , it follows, by Theorem 187, that DO is normal to BC .

Now, since AD is normal to P , it follows that AD is normal to BC .

Thus BC is normal to the two intersecting general lines DO and AD and must therefore be normal to the general plane containing them and which we may call Q .

But AO lies in Q and so it follows that AO must be normal to BC ; as was to be proved.

Consider now the second part of the theorem.

If BA and AC are both optical lines, then A , B and C are three corners of an optical parallelogram of which O is the centre and so by definition it follows that AO is normal to BC .

We shall therefore suppose that:

$$(B, A) \equiv (A, C),$$

and accordingly that BA and AC are either both inertia lines or both separation lines.

Let the general line through B parallel to AC intersect the general line through C parallel to AB in the element D .

Then B , A , C , D are the corners of a general parallelogram in the acceleration plane P and since O is the mean of B and C it follows that AD must intersect BC in the element O .

Thus AD must be identical with AO and O must be the mean of A and D .

But now we clearly have

$$(B, D) \equiv (A, C),$$

and so we must have $(B, D) \equiv (B, A)$.

Thus by the first part of the theorem it follows that BO is normal to DA : that is AO is normal to BC .

Thus both parts of the theorem are proved.

It is evident that the two parts of this theorem form the analogue of Theorem 187.

When however BC is an optical line, no analogue exists as is shown by Theorem 202.

THEOREM 204.

If A , B and C be three distinct elements which lie in an acceleration plane P , but do not all lie in one general line and if O be an element in BC such that AO is normal to BC , then:

(1) *If BC be a separation line and if*

$$(A, B) \equiv (A, C),$$

or if AB and AC be both optical lines, the element O must be the mean of B and C .

(2) *If BC be an inertia line and if*

$$(B, A) \equiv (A, C),$$

or if BA and AC be both optical lines, the element O must be the mean of B and C .

The proofs of these two parts are quite analogous to the proof of Theorem 188, using the two parts of Theorem 203 in place of Theorem 187.

THEOREM 205.

If A_1, B_1, C_1 be the corners of a triangle in an acceleration plane P_1 and A_2, B_2, C_2 be the corners of a triangle in an acceleration plane P_2 , and if further B_1C_1 be normal to A_1C_1 and B_2C_2 be normal to A_2C_2 , then:

(1) If A_1C_1 and A_2C_2 be inertia lines and

$$(C_1, A_1) \equiv (C_2, A_2),$$

and if

$$(A_1, B_1) \equiv (A_2, B_2),$$

or if A_1B_1 and A_2B_2 be both optical lines we shall also have

$$(C_1, B_1) \equiv (C_2, B_2).$$

(2) If A_1C_1 and A_2C_2 be inertia lines and

$$(A_1, C_1) \equiv (C_2, A_2),$$

and if

$$(A_1, B_1) \equiv (B_2, A_2),$$

or if A_1B_1 and A_2B_2 be both optical lines we shall also have

$$(C_1, B_1) \equiv (C_2, B_2).$$

(3) If A_1C_1 and A_2C_2 be separation lines and

$$(C_1, A_1) \equiv (C_2, A_2),$$

and if

$$(A_1, B_1) \equiv (A_2, B_2),$$

or if A_1B_1 and A_2B_2 be both optical lines we shall also have either

$$(C_1, B_1) \equiv (C_2, B_2),$$

or

$$(B_1, C_1) \equiv (C_2, B_2).$$

(4) If A_1C_1 and A_2C_2 be separation lines and

$$(A_1, C_1) \equiv (C_2, A_2),$$

and if

$$(A_1, B_1) \equiv (B_2, A_2),$$

we shall also have either

$$(B_1, C_1) \equiv (C_2, B_2),$$

or

$$(C_1, B_1) \equiv (C_2, B_2).$$

Cases (3) and (4) become equivalent if A_1B_1 and A_2B_2 are both separation lines and we have then an ambiguity in the result.

If on the other hand A_1B_1 and A_2B_2 are both inertia lines the first mentioned alternatives hold in cases (3) and (4).

If we only know that A_1B_1 and A_2B_2 are optical lines we again have ambiguity.

The proof of part (1) of this theorem is quite analogous to that of Theorem 190, using Theorem 200 (1) in place of Theorem 189, and Theorem 203 (1) in place of Theorem 187.

Similarly the proof of part (2) is quite analogous to that of Theorem 190, using Theorem 200 (2) in place of Theorem 189, and Theorem 203 (1) in place of Theorem 187.

As regards part (3), this is clearly equivalent to part (4) if A_1B_1 and A_2B_2 are separation lines, for then

$$(A_2, B_2) \equiv (B_2, A_2),$$

and the proof is again analogous to that of Theorem 190, with a slight modification.

This modification is required because in this case B_1C_1 and B_2C_2 must be inertia lines and it is not possible to find an element B_2' in B_2C_2 and on the same side of C_2 as is B_2 such that:

$$(C_1, B_1) \equiv (C_2, B_2'),$$

unless either B_1 is *before* C_1 and B_2 *before* C_2 , or else B_1 is *after* C_1 and B_2 *after* C_2 .

If either of these conditions hold the proof is analogous to Theorem 190, using Theorem 200 (1) in place of Theorem 189, and Theorem 203 (2) in place of Theorem 187.

If, on the other hand, we have either B_1 *before* C_1 and B_2 *after* C_2 , or else B_1 *after* C_1 and B_2 *before* C_2 , we must take an element B_2' in B_2C_2 , and on the same side of C_2 as is B_2 , and such that:

$$(C_1, B_1) \equiv (B_2', C_2).$$

Then Theorem 200 (2) takes the place of Theorem 189, and Theorem 203 (2) takes the place of Theorem 187.

In the former of these cases we get

$$(C_1, B_1) \equiv (C_2, B_2),$$

and in the latter we get

$$(B_1, C_1) \equiv (C_2, B_2).$$

If (A_1, B_1) and (A_2, B_2) are inertia pairs, then in part (3) their congruence shows that either B_1 is *after* A_1 and B_2 *after* A_2 , or else

B_1 is before A_1 and B_2 before A_2 , and in these cases we can always find an element B_2' in B_2C_2 , on the same side of C_2 as is B_2 , and such that:

$$(C_1, B_1) \equiv (C_2, B_2').$$

Thus, by Theorem 200 (1), we have

$$(A_1, B_1) \equiv (A_2, B_2').$$

and so we get

$$(A_2, B_2) \equiv (A_2, B_2').$$

But, since B_2 and B_2' lie in an inertia line, it follows from the footnote on page 349, that they must be identical, and so we must have

$$(C_1, B_1) \equiv (C_2, B_2).$$

In part (4), if (A_1, B_1) and (B_2, A_2) are inertia pairs, their congruence shows that either A_1 is after B_1 and B_2 after A_2 or else A_1 is before B_1 and B_2 before A_2 , and then, using Theorem 200 (2), we may show in a similar manner that:

$$(B_1, C_1) \equiv (C_2, B_2).$$

If A_1B_1 and A_2B_2 are optical lines the various parts of the theorem follow directly from Theorem 181 and the remark appended to it.

THEOREM 206.

If A_1, B_1, C_1 be the corners of a triangle in an acceleration plane P_1 , and A_2, B_2, C_2 be the corners of a triangle in an acceleration plane P_2 , and if A_1C_1 be a separation line which is normal to the inertia line B_1C_1 , then:

$$(1) \text{ If } (A_1, C_1) \equiv (A_2, C_2),$$

$$(B_1, C_1) \equiv (B_2, C_2),$$

and if

$$(A_1, B_1) \equiv (A_2, B_2),$$

or if A_1B_1 and A_2B_2 be both optical lines, we must also have A_2C_2 normal to B_2C_2 .

$$(2) \text{ If } (A_1, C_1) \equiv (A_2, C_2),$$

$$(C_1, B_1) \equiv (B_2, C_2),$$

and if

$$(A_1, B_1) \equiv (B_2, A_2),$$

or if A_1B_1 and A_2B_2 be both optical lines, we must also have A_2C_2 normal to B_2C_2 .

From the congruences it follows that since A_1C_1 is a separation line, A_2C_2 must be a separation line, and, since B_1C_1 is an inertia line, B_2C_2 must be an inertia line.

Thus any general line normal to B_2C_2 must be a separation line.

The proof of part (1) is then quite analogous to that of Theorem 191, using Theorem 200 (1) in place of Theorem 189, and Theorem 203 in place of Theorem 187, while remembering Theorem 202 and footnote.

Similarly the proof of part (2) is analogous to that of Theorem 191, using Theorem 200 (2) in place of Theorem 189, and Theorem 203 in place of Theorem 187, again remembering Theorem 202 and footnote.

Thus both parts of the theorem hold.

It is easy to show that for the case of acceleration planes there are also theorems analogous to Theorems 192 and 193, but it is to be observed that optical lines are exceptional since the only congruence of optical pairs is co-directional congruence.

It will be found however that this does not present much difficulty so far as the geometry is concerned.

ANALOGUE OF THE THEOREM OF PYTHAGORAS IN AN ACCELERATION PLANE.

We have seen that the geometry of a separation plane is formally identical with that of an ordinary Euclidean plane, and accordingly in a separation plane the theorem of Pythagoras connecting the sides of a right-angled triangle must hold.

Consider now the constructions for the two cases of Theorem 200 merely as regards the triangle whose corners are A_1 , B_1 , C_1 .

In case (1) B_1C_1 is a separation line, A_1C_1 is an inertia line normal to B_1C_1 , and A_1B_1 is a separation line.

But now we obtained a triangle whose corners were B_1 , C_1 and E_1 which lay in the separation plane S_1 and such that E_1B_1 was normal to E_1C_1 and in which accordingly we must have the segment relation:

$$(B_1C_1)^2 = (E_1B_1)^2 + (E_1C_1)^2.$$

This triangle was related to the one whose corners are A_1 , B_1 , C_1 in such a way that:

$$(E_1, B_1) \equiv (B_1, A_1),$$

while (C_1, E_1) was a before- or after-conjugate to (C_1, A_1) .

Thus taking segments instead of pairs we get

$$(B_1C_1)^2 = (B_1A_1)^2 + (\text{conjugate } C_1A_1)^2.$$

Thus the analogue of the theorem of Pythagoras is in this case:

$$(B_1A_1)^2 = (B_1C_1)^2 - (\text{conjugate } C_1A_1)^2 \dots\dots\dots(i).$$

Again if we consider case (2) we have B_1C_1 is a separation line, A_1C_1 is an inertia line normal to B_1C_1 , and A_1B_1 is also an inertia line.

In this case we obtained a triangle whose corners were C_1 , B_1 and F_1 which lay in the separation plane S_1 and such that B_1F_1 was normal to B_1C_1 .

Thus we must have the segment relation :

$$(C_1F_1)^2 = (B_1C_1)^2 + (B_1F_1)^2.$$

This triangle was related to the one whose corners are A_1 , B_1 , C_1 in such a way that (C_1, F_1) was a before- or after-conjugate to (C_1, A_1) while (B_1, F_1) was a before- or after-conjugate to (B_1, A_1) , and so, taking segments instead of pairs, we get

$$(\text{conjugate } C_1A_1)^2 = (B_1C_1)^2 + (\text{conjugate } B_1A_1)^2.$$

Thus the analogue of the theorem of Pythagoras is in this case :

$$- (\text{conjugate } B_1A_1)^2 = (B_1C_1)^2 - (\text{conjugate } C_1A_1)^2 \dots\dots(ii).$$

In the case where A_1B_1 is an optical line we obviously have

$$0 = (B_1C_1)^2 - (\text{conjugate } C_1A_1)^2 \dots\dots\dots(iii).$$

Thus (i), (ii) and (iii) constitute the complete analogue of the Pythagorean theorem in an acceleration plane.

If we consider a triangle whose corners are A_1 , B_1 , C_1 and which lies in an optical plane, then if B_1C_1 be a separation line and A_1C_1 be normal to B_1C_1 we know that A_1C_1 must be an optical line, while A_1B_1 must be another separation line.

Now we have shown that :

$$(B_1, A_1) \equiv (B_1, C_1),$$

and so taking segments instead of pairs we see that :

$$(B_1A_1)^2 = (B_1C_1)^2 \dots\dots\dots(iv).$$

This is the analogue of the Pythagorean theorem in an optical plane.

Considering now equations (i), (ii), (iii) and (iv) we observe that *the modifications which take place in the theorem of Pythagoras are such that when any side of the triangle becomes an inertia segment the corresponding square is replaced by the negative square of the conjugate of this inertia segment, while if any side becomes an optical segment, the corresponding square is replaced by zero.*

Again if A_1 , B_1 , C_1 be the corners of a triangle whose sides are of the stated kinds and for which one of the relations (i), (ii), (iii) or (iv) holds, it is easy to show that B_1C_1 must be normal to A_1C_1

Take for instance the case where B_1C_1 is a separation line, A_1C_1 an inertia line and A_1B_1 a separation line, and where

$$(B_1A_1)^2 = (B_1C_1)^2 - (\text{conjugate } C_1A_1)^2.$$

Let A_2, B_2, C_2 be the corners of another triangle in an acceleration plane, such that B_2C_2 is a separation line normal to the inertia line A_2C_2 , and where

$$(B_2, C_2) \equiv (B_1, C_1),$$

and

$$(C_2, A_2) \equiv (C_1, A_1).$$

Then the segment B_2C_2 must be greater than the conjugate to the segment C_2A_2 , from which it is easy to see that B_2A_2 must be a separation line.

Thus we must have

$$(B_2A_2)^2 = (B_2C_2)^2 - (\text{conjugate } C_2A_2)^2,$$

and so we must also have

$$(B_2, A_2) \equiv (B_1, A_1).$$

Then by Theorem 206, since B_2C_2 is normal to A_2C_2 , it follows that B_1C_1 is normal to A_1C_1 .

The cases where A_1B_1 is an inertia line and relation (ii) holds and where A_1B_1 is an optical line and relation (iii) holds may be treated in a similar manner, and we may prove that B_1C_1 is normal to A_1C_1 .

Again, if A_1, B_1, C_1 be the corners of a triangle in which B_1C_1 and B_1A_1 are separation lines while A_1C_1 is an optical line, and if

$$(B_1A_1)^2 = (B_1C_1)^2,$$

then

$$(B_1, A_1) \equiv (B_1, C_1),$$

and so, as was pointed out in the remarks at the end of Theorem 202, the elements A_1, B_1, C_1 must lie in an optical plane.

Thus since A_1C_1 is an optical line, it follows that B_1C_1 is normal to A_1C_1 .

Let A, B, C be the corners of a triangle in an acceleration plane P , and let AB, BC and CA be all separation lines or all inertia lines.

It is easy to see that triangles of both these kinds exist, although as Theorem 31 shows it is not possible for AB, BC and CA to be all optical lines.

Let a_1, b_1, c_1 be generators of P of one set, which pass through A, B, C respectively and intersect BC, CA, AB in A_1, B_1, C_1 respectively.

Then we may show by a method similar to that employed in the

remarks at the end of Theorem 199, that one and only one of the elements A_1, B_1, C_1 is linearly between a pair of the corners A, B, C .

Similarly we may show that if a_2, b_2, c_2 be generators of P of the opposite set passing through the elements A, B, C respectively and intersecting BC, CA, AB in A_2, B_2, C_2 respectively, then one and only one of the elements A_2, B_2, C_2 is linearly between a pair of the corners A, B, C .

Now consider the case, for instance, where A_1 is linearly between B and C , and suppose first that AB, BC, CA are all separation lines.

Then B cannot be linearly between A_1 and A_2 for then, by Theorem 129 (a) and (b), AB would require to be an inertia line, contrary to hypothesis.

Similarly C cannot be linearly between A_1 and A_2 .

Thus, since obviously A_2 cannot be identical with either B or C , it follows that A_2 must be also linearly between B and C .

Now let O be the mean of A_1 and A_2 .

Then O is linearly between A_1 and A_2 , and therefore clearly it must lie linearly between B and C .

But now A_1, A, A_2 are three corners of an optical parallelogram of which O is the centre, and so AO must be normal to A_1A_2 : that is to BC .

Again, if instead of AB, BC, CA being all separation lines they are all inertia lines, a similar result holds.

Let us take the case where A_1 is linearly between B and C .

Then clearly B cannot be linearly between A_1 and A_2 , for then AB would require to be a separation line, and, for a similar reason, C cannot be linearly between A_1 and A_2 .

Thus, since A_2 cannot coincide with either B or C , it follows that A_2 must also be linearly between B and C .

As in the former case, if O be the mean of A_1 and A_2 , then O must be linearly between B and C , and AO must be normal to BC .

Now in the case where AB, BC, CA are all separation lines, it follows from relation (i) that

segment BA is less than segment BO ,

and

segment AC is less than segment OC .

Thus it follows that the sum of the lengths of the segments BA and AC is less than that of the segment BC .

Similarly, if AB, BC, CA be all inertia lines, we may show by the help of relation (ii) that the sum of the lengths of the segments BA and AC is less than that of the segment BC .

Now we know that in ordinary Euclidean geometry the sum of the lengths of any two sides of a triangle is greater than that of the third, and a similar result must hold in a separation plane.

Thus, remembering what was proved at the end of Theorem 199, we have the following interesting results:

If A, B, C be the corners of a general triangle all whose sides are segments of one kind, then:

(1) *If the triangle lies in a separation plane, the sum of the lengths of any two sides is greater than that of the third side.*

(2) *If the triangle lies in an optical plane, the sum of the lengths of a certain two sides is equal to that of the third side.*

(3) *If the triangle lies in an acceleration plane, the sum of the lengths of a certain two sides is less than that of the third side.*

If A_1, B_1, C_1 be the corners of a triangle such that A_1B_1, B_1C_1, C_1A_1 are all separation lines, and if

$$(B_1A_1)^2 = (B_1C_1)^2 + (C_1A_1)^2,$$

it is evident that the sum of the lengths of any two sides must be greater than that of the third side.

Thus it follows from what we have just shown that such a triangle cannot lie in any type of general plane, except a separation plane.

In this case, however, we know from ordinary geometry that we must have B_1C_1 normal to A_1C_1 .

On the other hand, since an inertia line cannot lie in any but an acceleration plane, it is not possible to have A_1B_1, B_1C_1, C_1A_1 all inertia lines and also to have the relation

$$(B_1A_1)^2 = (B_1C_1)^2 + (C_1A_1)^2.$$

Having thus investigated the analogues of the theorem of Pythagoras and its converse, we are now in a position to introduce coordinates.

INTRODUCTION OF COORDINATES.

If we take any element O of the set as origin, we have already seen that we may obtain systems of four general lines through O , say OX, OY, OZ, OT , which are mutually normal to one another.

Three of these, say OX, OY, OZ , will be separation lines, while the fourth, OT , will be an inertia line.

The three separation lines OX, OY, OZ will determine a separation threefold, say W , and OT will be normal to it.

If we select any arbitrary separation segment as a unit of length and associate the number zero with the element O , we may associate every other element of OX , OY , OZ with a real number, positive or negative, corresponding to the length of the segment of which that element is one end and the origin O is the other.

In this way we set up a coordinate system in W which will be quite similar to that with which we are familiar.

Since all the theorems of ordinary Euclidean geometry hold for a separation threefold, the length of a segment in W will be given by the ordinary Cartesian formula.

Again, not confining our attention merely to the elements of W , let A be any element of the whole set.

Then A must either lie in OT , or else there is an inertia line through A parallel to OT , and, as has already been proved, this inertia line will intersect W in some element, say N .

Further, AN must be normal to W .

Now if A does not lie in W there will be a separation threefold, say W' , passing through A and parallel to W , and the inertia line OT must intersect W' in some element, say M .

Further, since W' is parallel to W , both OT' and AN must be normal to W' .

Thus, if OM and NA are distinct, MA and ON must both be separation lines normal to OM , and so, since OM and NA lie in an acceleration plane, we must have MA parallel to ON .

Now we may select a unit inertia segment, just as we selected a unit separation segment, and with each element of OT distinct from O we may associate a real number positive or negative corresponding to the length of the segment of which that element is one end and the origin O is the other.

We shall suppose this correspondence to be set up in such a way that a positive real number corresponds to any element which is after O and a negative real number to any element which is before O .

As regards the relationship between the unit separation segment and the unit inertia segment, the simplest convention to make is to take the unit inertia segment such that its conjugate is equal to the unit separation segment.

More generally, we may take the unit inertia segment such that:

$$(\text{conjugate of unit inertia segment}) = v (\text{unit separation segment}),$$

where v is a constant afterwards to be identified with what we call the "velocity of light."

Now the element N lies in W and is determined by three co-ordinates, say x_1, y_1, z_1 , taken parallel to OX, OY, OZ respectively in the usual manner.

Further segment $NA = \text{segment } OM$,

and so if t_1 be the length of OM in terms of the unit inertia segment, then the element A will be determined by the four coordinates x_1, y_1, z_1, t_1 .

Let the length of the segment ON be denoted by a .

Then as in ordinary coordinate geometry

$$a^2 = x_1^2 + y_1^2 + z_1^2.$$

Thus if OA should be an optical line, we must have

$$a^2 = v^2 t_1^2,$$

or
$$x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2 = 0 \dots \dots \dots (1).$$

Again, if OA should be a separation segment and if r_1 be its length, it follows from the analogue of the theorem of Pythagoras for this case that:

$$a^2 - v^2 t_1^2 = r_1^2,$$

or
$$x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2 = r_1^2 \dots \dots \dots (2).$$

Finally, if OA should be an inertia segment and \bar{r}_1 its length, it follows from the corresponding analogue of the Pythagoras theorem that:

$$a^2 - v^2 t_1^2 = -v^2 \bar{r}_1^2,$$

or
$$x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2 = -v^2 \bar{r}_1^2 \dots \dots \dots (3).$$

Thus from (1), (2) and (3) it follows that the expression

$$x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2$$

is positive, zero, or negative according as OA is a separation line, an optical line, or an inertia line.

If A be after O , it is clear from the convention which we have made that t_1 must be positive, and so the conditions that A should be after O are:

$$\left. \begin{array}{l} (1) \quad x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2 \text{ is zero or negative} \\ (2) \quad t_1 \text{ is positive} \end{array} \right\}.$$

The conditions that A should be before O are similarly:

$$\left. \begin{array}{l} (1) \quad x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2 \text{ is zero or negative} \\ (2) \quad t_1 \text{ is negative.} \end{array} \right\}.$$

Thus if (x_0, y_0, z_0, t_0) be the coordinates of an element A_0 the equation of the combined α and β sub-sets of A_0 is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - v^2(t - t_0)^2 = 0 \dots\dots\dots(7).$$

The α sub-set of A_0 will then consist of all elements (x, y, z, t) for which this equation is satisfied and for which $t - t_0$ is zero or positive; while the β sub-set of A_0 will consist of all elements for which the equation is satisfied and for which $t - t_0$ is zero or negative.

Definition. The set of all elements whose coordinates satisfy equation (7) will be called the *standard cone* with respect to the element whose coordinates are (x_0, y_0, z_0, t_0) .

Taking v equal to unity, for the sake of simplicity, it is evident that the equation

$$x^2 + y^2 + z^2 - t^2 = c^2$$

represents the set of elements such as A , where OA is a separation segment whose length is c .

Similarly, the equation

$$x^2 + y^2 + z^2 - t^2 = -c^2$$

represents the set of elements such as A , where OA is an inertia segment whose length is c .

If we put $y = 0$ and $z = 0$ in the first of these we obtain

$$x^2 - t^2 = c^2,$$

which gives us the relation between x and t for the portion of the corresponding set which lies in the acceleration plane containing the axes of x and t .

This then represents the analogue of a circle in the acceleration plane.

Similarly for the case of inertia segments putting $y = 0$ and $z = 0$ we get

$$x^2 - t^2 = -c^2.$$

The two equations

$$x^2 - t^2 = c^2 \quad \text{and} \quad x^2 - t^2 = -c^2$$

are of the same forms as the equations of a hyperbola and its conjugate in ordinary plane geometry.

The equation

$$x^2 - t^2 = 0$$

along with $y = 0$ and $z = 0$ represents the two optical lines through the origin in the same acceleration plane, and these correspond to the common asymptotes of the hyperbolas.

EQUATIONS OF AN OPTICAL LINE.

For the sake of simplicity we shall again take v equal to unity.

Let (x_1, y_1, z_1, t_1) be the coordinates of an element A_1 and let (x_2, y_2, z_2, t_2) be the coordinates of an element A_2 distinct from A_1 and lying either in the α or β sub-set of A_1 .

The equation of the standard cone with respect to the element A_1 is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 - (t - t_1)^2 = 0 \dots\dots\dots(1).$$

Since A_2 lies either in the α or β sub-set of A_1 we must have

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - (t_2 - t_1)^2 = 0 \dots\dots\dots(2).$$

If A_2 lies in the α sub-set of A_1 we must have $t_2 - t_1$ positive, while if A_2 lies in the β sub-set of A_1 we must have $t_2 - t_1$ negative.

The equation of the standard cone with respect to the element A_2 is

$$(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 - (t - t_2)^2 = 0 \dots\dots\dots(3).$$

Remembering now the definition of an optical line given on page 13 and observing that if A_1 and A_2 be distinct elements and if A_2 lies in α_1 there can be no element common to α_2 and β_1 , we see that any element of the optical line defined by A_1 and A_2 must be such that its coordinates satisfy equations (1) and (3).

Further the set of elements whose coordinates satisfy (1) and (3) must be identical with the optical line containing A_1 and A_2 provided equation (2) holds.

Adding equations (1) and (2) and subtracting equation (3) we get on dividing by 2

$$(x_2 - x_1)(x - x_1) + (y_2 - y_1)(y - y_1) + (z_2 - z_1)(z - z_1) - (t_2 - t_1)(t - t_1) = 0.$$

Thus

$$\{(x_2 - x_1)(x - x_1) + (y_2 - y_1)(y - y_1)\}^2 \\ = \{(z_2 - z_1)(z - z_1) - (t_2 - t_1)(t - t_1)\}^2 \dots\dots(4).$$

But from (1) and (2) we get

$$\{(x - x_1)^2 + (y - y_1)^2\} \{(x_2 - x_1)^2 + (y_2 - y_1)^2\} \\ = \{(z - z_1)^2 - (t - t_1)^2\} \{(z_2 - z_1)^2 - (t_2 - t_1)^2\} \dots\dots(5).$$

Thus from equations (4) and (5) we obtain

$$\{(y - y_1)^2(x_2 - x_1)^2 - 2(y - y_1)(x_2 - x_1)(x - x_1)(y_2 - y_1) + (x - x_1)^2(y_2 - y_1)^2\} \\ = -\{(t - t_1)^2(z_2 - z_1)^2 - 2(t - t_1)(z_2 - z_1)(z - z_1)(t_2 - t_1) + (z - z_1)^2(t_2 - t_1)^2\}.$$

Thus for all real values of the coordinates we must have

$$(y - y_1)(x_2 - x_1) - (x - x_1)(y_2 - y_1) = 0,$$

and

$$(t - t_1)(z_2 - z_1) - (z - z_1)(t_2 - t_1) = 0.$$

By a similar method we find that :

$$(z - z_1)(y_2 - y_1) - (y - y_1)(z_2 - z_1) = 0,$$

and

$$(t - t_1)(x_2 - x_1) - (x - x_1)(t_2 - t_1) = 0.$$

Thus all elements which lie in the optical line passing through A_1 and A_2 must be such that their coordinates (x, y, z, t) satisfy the equations

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \frac{t - t_1}{t_2 - t_1} \dots\dots\dots(6).$$

Conversely a set of equations of the form (6) will represent an optical line, provided that :

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - (t_2 - t_1)^2 = 0.$$

The expression in coordinate form of the various results which we have obtained by geometrical methods is now merely a matter of straightforward analysis.

In carrying out this analysis (as we have just seen in connection with the equations of an optical line) the fact that the coordinates of any element must be *real* frequently plays an important part.

Since our main object has been to show how a system of geometry may be built up from ideas of *after* and *before*, it is unnecessary to go into these matters in detail in the present volume.

INTERPRETATION OF RESULTS.

It is evident that any element whose coordinates are $(a, b, c, 0)$ must lie in the separation threefold W and accordingly the three equations

$$x = a, \quad y = b, \quad z = c$$

must represent an inertia line normal to W and therefore parallel to or identical with the axis of t .

Again, any equation of the first degree in x, y, z , together with the equation $t = 0$, will represent a separation plane in W , while any two independent but consistent equations of the first degree in x, y, z , together with the equation $t = 0$, will represent a separation line in W .

Thus any equation of the first degree in x, y, z (leaving out the equation $t = 0$) will represent a rotation threefold containing inertia

lines parallel to the axis of t ; while any two independent but consistent equations of the first degree in x, y, z will represent an acceleration plane containing inertia lines parallel to the axis of t .

Thus corresponding to any theorem concerning the elements of W there will be a theorem concerning inertia lines normal to W and passing through these elements.

Conversely, if we consider the system consisting of any selected inertia line together with all others parallel to it, then any two such inertia lines will determine an acceleration plane, while any three which do not lie in one acceleration plane will determine a rotation threefold.

Since these inertia lines must all intersect any separation threefold to which they are normal, it follows that they have a geometry similar to that of the separation threefold and therefore of the ordinary Euclidean type.

If then we call any element of the entire set an "instant"; any inertia line of the selected system a "point"; any acceleration plane of the selected system a "straight line"; and any rotation threefold of the selected system a "plane"; we can speak of succeeding instants at any given point, and have thus obtained a representation of the space and time of our experience in so far as their geometrical relations are concerned.

The distance between two parallel inertia lines of the system will naturally be taken as the length of the segment intercepted by them in a separation line which intersects them both normally.

This, then, will be the meaning to be attached to the *distance between two points*.

Time intervals in the usual sense will be measured by the lengths of segments of the corresponding inertia lines: that is to say, by differences of the t coordinates.

Since we have defined the equality of separation and inertia segments in terms of the relations of *after* and *before* and have assigned an interpretation to these, it follows that the equality of length and time intervals in the ordinary sense is rendered precise.

It is to be observed that the particular system of parallel inertia lines which we may select is quite arbitrary although the set of elements or instants contained in the entire system is in all cases identical.

The distinction between different systems is that while two parallel inertia lines represent the time paths of unaccelerated particles which are at rest relative to one another; two non-parallel inertia lines represent the time paths of unaccelerated particles which are in motion with uniform velocity with respect to one another.

Thus we are able to give a definition of absence of acceleration, but,

since all inertia lines are on a par with one another, we can attach no meaning to a particle or system being at "absolute rest."

The definition of absence of acceleration based upon the relations of *after* and *before* and as regards a finite interval of time, may be thus expressed :

Definition. If A and B be two distinct elements of any inertia line (B being *after* A), then a particle will be said to be *unaccelerated from the instant A to the instant B* provided it lies in the inertia line AB throughout that interval.

The physical signification of an optical line is: that a flash of light or other instantaneous electromagnetic disturbance in going directly from one particle to another would follow this time path.

As regards a separation line; since no element of it is either *before* or *after* another, then if our view be correct, no single particle could occupy more than one element, and so particles which occupy distinct elements of any separation line must be separate particles.

The above considerations indicate the reasons for adopting the names we have assigned to the three types of general line.

We shall suppose that the time path of any particle is a continuous curve having at every element a tangent which is an inertia line.

The velocity of any particle with respect to a system of coordinates such as we have described will, then, in all cases be limited by what we call the "velocity of light."

Any acceleration of the particle will thus determine an acceleration plane; that being the type of general plane which osculates the time path.

Similarly if the acceleration plane varies from element to element of the time path so that the latter is tortuous, then this tortuosity determines a rotation threefold.

The names optical plane and optical threefold have been adopted because of the analogies of these to optical lines, and similarly the names separation plane and separation threefold have been adopted on account of the analogies to separation lines.

Results involving only three coordinates x , y and t may be visualized by means of the three-dimensional conical order described in the introduction, but a certain amount of distortion appears in a model of this kind, since equal lengths in the model do not in general represent equal lengths as we have defined them.

The optical significations of Posts. I to XVIII are however made clear by such models, and it is easily seen that the assertions made in

these postulates, when interpreted in the manner described, are in accordance with the ordinarily accepted ideas.

Post. XXI also finds an interpretation in such a model, but its significance is concerned rather with the logic of continuity than with any observable physical phenomenon.

Since it is possible to define equality of lengths in terms of *after* and *before* it seems superfluous to introduce any other conception of length, since the effect of this would merely be to destroy the symmetry which otherwise exists.

It is further to be noted that the formal development of the theory of conical order does not in itself require that the α and β sub-sets should be determined by optical phenomena, but merely that there should exist some influence having the properties which we have ascribed to light.

Accordingly if it should be found hereafter that some other influence than light possessed these properties we should merely require to substitute this influence for light and interpret our results in terms of it.

CONCLUSION.

Our task now approaches completion.

We have shown how from some twenty-one postulates involving the ideas of *after* and *before* it is possible to set up a system of geometry in which any element may be represented by four coordinates x, y, z, t .

Three of these: x, y, z , correspond to what we ordinarily call space coordinates, while the fourth corresponds to time as generally understood.

Since however an element in this geometry corresponds to an instant, and bears the relations of *after* and *before* to certain other instants, it appears that the theory of space is really a part of the theory of time.

Of the postulates used: nineteen, namely I to XVIII and Post. XXI, may easily be seen to have an interpretation in three-dimensional geometry by making use of cones as described in the introduction.

It follows that if ordinary geometry be consistent with itself, these nineteen postulates must be consistent with one another.

Of the remaining two postulates, Post. XIX has the effect of introducing one more dimension, while Post. XX limits the number of dimensions to four.

Since by means of these we have been enabled to set up a coordinate system in the four variables x, y, z, t , the question of the consistency of the whole twenty-one postulates is reduced to analysis.

It is not proposed to go further into this matter in the present volume, having said sufficient to leave little doubt that they are all consistent with one another.

The question as to whether the postulates are all independent is mainly a matter of logical nicety and is of comparatively little importance provided that the number of redundant postulates be not large.

In the course of development of the present work the writer succeeded in eliminating a considerable number of postulates which he had provisionally laid down: the redundancies being generally indicated by the possibility of proving some particular result from several sets of postulates.

One known redundancy has been permitted to remain: namely Post. II (a) and (b), which might have been deduced directly from Post. V and Post. VI (a) and (b).

By retaining Post. II however, our first four postulates will be seen to hold for the set of instants of which any one individual is directly conscious, and the subject is thus better exhibited as an extension of the commonly accepted ideas of time.

A still further diminution of the number of postulates might have been made by combining Posts. VI and XI in the way mentioned on page 25, but to have done so would have complicated still further the initial part of the subject, since Post. VI implies merely a two-dimensional conical order, while its combination with Post. XI makes the set of elements at least three-dimensional from the very beginning.

Apart from the above-mentioned, no further definite indications of redundancy have been observed, and, although some redundant postulates may still remain, it seems unlikely that there can be many.

This opinion is confirmed by a comparison with the number of fundamental assumptions given by various writers on the foundations of ordinary geometry.

* We have now concluded the exposition of the argument by which we have been led to the view expressed in the introduction: that *spacial relations are to be regarded as the manifestation of the fact that the elements of time form a system in conical order: a conception which may be analysed in terms of the relations of after and before.*

This view would appear to have important bearings on general philosophy, but into these we do not purpose here to enter.

One point may however be mentioned:

The fundamental properties of time must, on any theory, be regarded as possessing a character which is not transitory, but in some sense

persistent; since otherwise, statements about the past or future would be meaningless.

We here touch on the difficult problem as to the nature of "universals": a problem which has been much discussed by philosophers, but appears to be still far from a satisfactory solution.

Though space may be analyzable in terms of time relations, yet these remain in their ultimate nature as mysterious as ever; and though events occur in time, yet any logical theory of time itself must always imply the Unchangeable.

Thus may we conclude in the words of Carlyle:

"Know of a truth that only the Time-shadows have perished, or are perishable; that the real Being of whatever was, and whatever is, and whatever will be, *is* even now and forever."

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